Tutte polynomials in combinatorics and geometry

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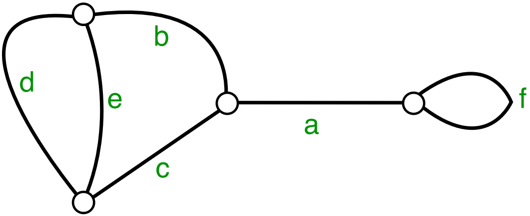
Outline

1. Motivating examples
2. Matroids
3. Corte de comerciales.
4. Tutte polynomials
5. Hyperplane arrangements
6. Computing Tutte polynomials

MOTIVATING EXAMPLES: 1. Graph Theory.

Goal: Build internet connections that will connect the 4 cities.

To lower costs, build the minimum number of connections.



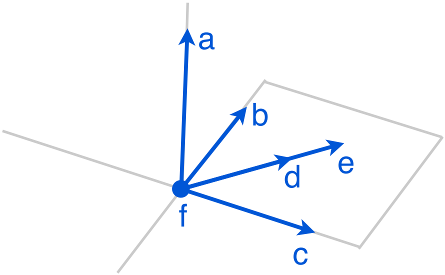
Solutions: $\{abc, abd, abe, acd, ace\}$

(The **spanning trees** of the graph.)

MOTIVATING EXAMPLES: 2. Linear Algebra.

Goal: Choose a minimal set of vectors that spans $\mathbb{R}^3$.

No 3 on a plane, no 2 on a line, no 1 at the origin.



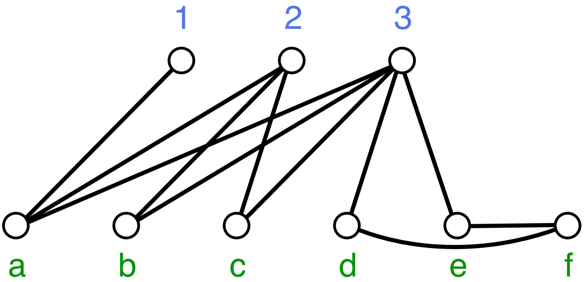
Solutions: $\{abc, abd, abe, acd, ace\}$

(The **bases** of the vector configuration.)

MOTIVATING EXAMPLES: 3. Matching Theory.

Goal: Marry as many people as possible.

No gay marriage in Texas.(!) No poligamy.



Possible married men: $\{abc, abd, abe, acd, ace\}$

(The **systems of distinct representatives**.)

MOTIVATING EXAMPLES: 4. Field Extensions.

Goal: Choose a transcendence basis for $\mathbb{C}[x, y, z]$ over $\mathbb{C}$.

Maximal set with no algebraic relations with coeffs. in $\mathbb{C}$.

$$\begin{aligned} a &= z^3 \\ b &= x + y \\ c &= x - y \\ d &= xy \\ e &= x^2y^2 \\ f &= 1 \end{aligned}$$

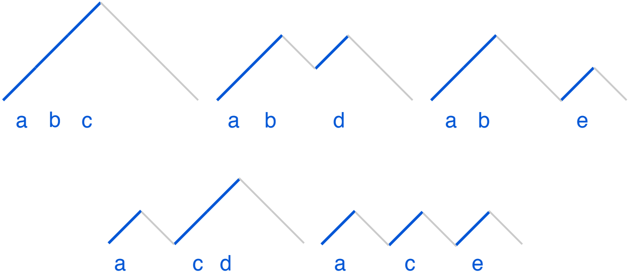
Solutions: $\{abc, abd, abe, acd, ace\}$

(The **transcendence bases** of the field extension.)

MOTIVATING EXAMPLES: 5. Catalan combinatorics.

Goal: Choose up-steps to get to $(6, 0)$ staying above the x-axis.

Never cross the x-axis.



Solutions: $\{abc, abd, abe, acd, ace\}$

(The **Dyck paths** of length 6.)

MATROIDS.

Definition. [MacLane / Nakasawa / Whitney 1930s]
A **matroid** $(E, \mathcal{B})$ consists of:

- A *ground set* E, and
- A collection $\mathcal{B}$ of subsets of E called **bases** such that:

If A and B are bases and $a \in A \setminus B$, then
there exists $b \in B \setminus A$ such that $A \setminus \{a\} \cup \{b\}$ is a basis.

Example. $E = \{a, b, c, d, e, f\}$ $\quad \mathcal{B} = \{abc, abd, abe, acd, ace\}$

Proposition. The 5 examples above give 5 families of matroids.

Proof. This "basis exchange axiom" holds in graph theory, linear algebra, matching theory, field extension theory, Catalan theory.

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So a theorem in matroid theory gives us theorems in ≥ 5 areas!

For example:

Theorem. All the bases of a matroid have the same size.

Corollaries.

- All spanning trees of a graph have = number of edges. (**V-1**)
- All bases of a vector space have = size. (**Dimension**)
- All maxl sets of marriable men have = size. (**Matching #**)
- All transcendence bases of L/K have = size. (**Transc. deg.**)
- All Dyck paths of length $2n$ have = number of up-steps. (**n**)

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A theorem in matroid theory gives us theorems in ≥ 5 areas!

Theorem. If $M = (E, \mathcal{B})$ is a matroid, then $M^* = (E, \mathcal{B}^*)$ is the **dual matroid**, where

$$\mathcal{B}^* = \{E \setminus B : B \text{ is a basis of } M\}$$

Examples.

If $E = \{a, b, c, d, e, f\}$
and $\mathcal{B} = \{abc, abd, abe, acd, ace\}$.

then $\mathcal{B}^* = \{def, cef, cdf, bef, bdf\}$.

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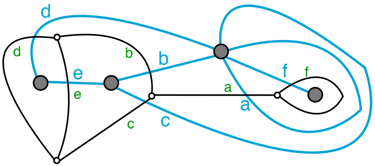
Examples. GRAPHS.

• If M is the matroid of a **planar** graph G, then M^* is the matroid of the dual graph G^*.



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Examples. VECTORS.

Say M is the matroid of vectors $\{a_1, \ldots, a_n\}$.
Write them as column vectors in $\mathbb{R}^d$ and let

$$A = \text{rowspace}[a_1 \, a_2 \ldots a_n],$$

and choose $b_1, \ldots, b_n$ in $\mathbb{R}^{n-d}$ so that

$$A^\perp = \text{rowspace}[b_1 \, b_2 \ldots b_n]$$

Then M^* is the matroid of $\{b_1, \ldots, b_n\}$.

So a theorem in matroid theory gives us theorems in ≥ 5 areas!

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Examples. MATCHINGS.

Unfortunately, if M is the matroid of a matching problem,
M^* is not necessarily the matroid of a matching problem!

Fortunately,
M^* is the matroid of a *routing problem* – a 6th kind of matroid.

So a theorem in matroid theory gives us theorems in ≥ 5 areas!

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Examples. FIELD EXTENSIONS.

If M is a matroid coming from elements of a field extension, noone knows whether M^* also comes from a field extension.

This problem is wide open!

So a theorem in matroid theory gives us theorems in ≥ 5 areas!

Theorem. If $M = (E, \mathcal{B})$ is a matroid, then $M^* = (E, \mathcal{B}^*)$ is the **dual matroid**, where

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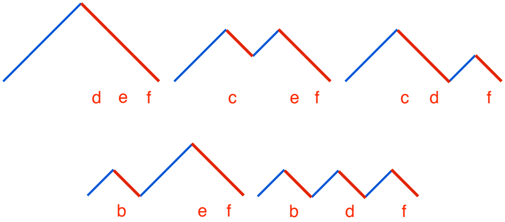
Examples. CATALAN.

If M is the Catalan matroid, then $M^* \cong M$:



$$\mathcal{B}^* = \{def, cef, cdf, bef, bdf\}.$$

Corte de comerciales.

San Francisco State University – Colombia Combinatorics Initiative

For more information on:

- enumerative combinatorics
- matroids,
- polytopes,
- Coxeter groups,
- combinatorial commutative algebra, and
- Hopf algebras in combinatorics

you may see the (200+) videos and lecture notes of my courses at San Francisco State University and the U. de Los Andes:

http://math.sfsu.edu/federico/
http://youtube.com/user/federicoelmatematico

THE TUTTE POLYNOMIAL.

Let $M = (E, \mathcal{B})$ be a matroid.

(If you prefer, think that E is a set of vectors in $\mathbb{R}^d$.)

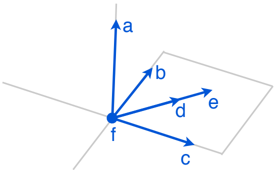
The **rank** of $A \subseteq E$ is

$$r(A) = \max_{B \text{ basis}} |A \cap B|;$$

that is, the size of the largest "independent" set in A.

Definition. [Tutte, 1967] The **Tutte polynomial** of $\mathcal{A}$ is

$$T_{\mathcal{A}}(x, y) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (x-1)^{r(\mathcal{A})-r(\mathcal{B})}(y-1)^{|\mathcal{B}|-r(\mathcal{B})}.$$



S	$\lvert S\rvert$	$r(S)$	$(x-1)^{r-r(S)}(y-1)^{\lvert S\rvert-r(S)}$
$\emptyset$	0	0	$(x-1)^3(y-1)^0$
$a\ b\ c\ d\ e$	1	1	$(x-1)^2(y-1)^0$
f	1	0	$(x-1)^3(y-1)^1$
$ab\ ac\ ad\ ae\ bc\ bd\ be\ cd\ ce$	2	2	$(x-1)^1(y-1)^0$
$af\ bf\ cf\ de\ df\ ef$	2	1	$(x-1)^2(y-1)^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} T(x,y) &= (x-1)^3+5(x-1)^2+(x-1)^3(y-1)+9(x-1)+\cdots \\ &= x^3y+x^2y+x^2y^2+xy^2+xy^3 \end{aligned}$$

Clearly there is something more to this story...

WHY CARE ABOUT THE TUTTE POLYNOMIAL?

Many interesting quantities are evaluations of $T_{\mathcal{A}}(x, y)$.

For graphs:

- $T(1,1)$ = number of **spanning trees**.
- $T(2,0)$ = number of **acyclic orientations** of edges.
- $T(0,2)$ = number of **totally cyclic orientations** of edges.
- $(-1)^{v-c} q^c T(1-q, 0)$ = **chromatic polynomial** = number of proper q-colorings of the vertices.
- $(-1)^{e-v+c} T(0, 1-t)$ = **flow polynomial** = number of nowhere zero t-flows of the edges.

[Tutte, 1947] [Crapo, 1969] [Stanley, 1973] [LasVergnas, 1980]

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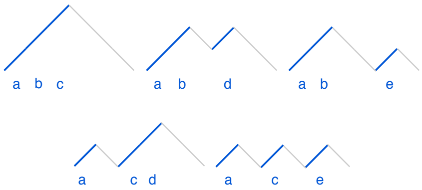
[Tutte, 1947] [Crapo, 1969] [Stanley, 1973] [LasVergnas, 1980]

Many interesting invariants of $\mathcal{A}$ are evaluations of $T_{\mathcal{A}}(x, y)$.

For the Catalan matroid: [A. 02]

- $T(1,1) = \frac{1}{n+1}\binom{2n}{n}$ (Catalan numbers)
- If $a(P)$ = number of up-steps before the first down-step, and $b(P)$ = number of returns to the x-axis.

$$T(x,y) = \sum_{P \text{ Dyck}} x^{a(P)} y^{b(P)}$$



$$T(x,y) = x^3y + x^2y + x^2y^2 + xy^2 + xy^3$$

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For the Catalan matroid: For a path P let
$a(P)$ = number of up-steps before the first down-step,
$b(P)$ = number of times the path bounces on the x-axis.

$$T(x,y) = \sum_{P \text{ Dyck}} x^{a(P)} y^{b(P)}$$

Theorem. The Tutte polynomials of M and M^* are related by

$$T_{M^*}(x,y) = T_M(y,x).$$

Since $C_n^* \cong C_n$ we get $T_{C_n}(x,y) = T_{C_n}(y,x)$, so

Theorem. [A. 2002]
(# of Dyck paths of $2n$ steps, a initial upsteps, b bounces) =
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For vector arrangements:

- $T(1, 1)$ = number of **bases**.
- $T(2, 1)$ = number of **independent sets**.
- $T(1, 2)$ = number of **spanning sets**.

Many interesting invariants of $\mathcal{A}$ are evaluations of $T_{\mathcal{A}}(x, y)$.

For vector arrangements $\mapsto$ hyperplane arrangements:

Vector $a \in \mathbb{K}^n \mapsto$ Hyperplane $H_a = \{x \in (\mathbb{K}^n)^* \, : \, a \cdot x = 0\}$.

Vector arr. $A \subseteq \mathbb{K}^n \mapsto$ Complement $V(\mathcal{A}) = \mathbb{K}^n \setminus$ hyperplanes

Example. C_3

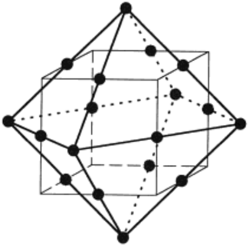
Vectors:

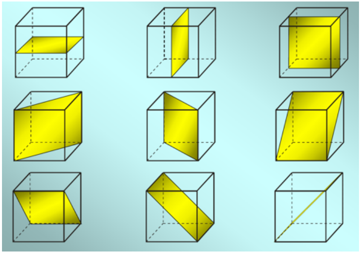
- $\pm e_i \quad (1 \leq i \leq 3)$
- $\pm e_i \pm e_j \quad (1 \leq i < j \leq 3)$

Hyperplanes:

$2x = 0, 2y = 0, 2z = 0$

$x \pm y = 0, y \pm z = 0, z \pm x = 0$





Many important invariants of $\mathcal{A}$ are evaluations of $T_{\mathcal{A}}(x, y)$.

For hyperplane arrangements:

- $(\mathbb{K} = \mathbb{R})$

$$(-1)^n T(2,0) = \text{number of regions of } V(\mathcal{A})$$

[Zaslavsky, 1975]

- $(\mathbb{K} = \mathbb{C})$

$$T(1-q,0) = \sum_i \dim H^i(V(A);\mathbb{Z})(-q)^i$$

[Orlik and Solomon, 1980, Goresky-MacPherson, 1988]

- $(\mathbb{K} = \mathbb{F}_q)$

$$|T(1-q,0)| = |V(\mathcal{A})|$$

[Crapo and Rota, 1970]

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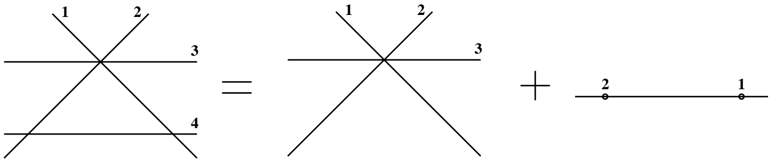
WHY IS THE TUTTE POLYNOMIAL IN SO MANY PLACES?

Given a matroid M and an element e:

Deletion: $M \setminus e$ has bases $\{B \in \mathcal{B} \,:\, e \notin B\}$
Contraction: M/e has bases $\{B - e \,:\, B \in \mathcal{B}, e \in B\}$

A **Tutte-Grothendieck** invariant is a function which behaves well under deletion and contraction:

$$f(M) = f(M \setminus e) + f(M/e) \qquad \text{(for all nontrivial } e\text{)}$$



Theorem. (Brylawski, 1972) The Tutte polynomial is the universal T-G invariant. Every other one is an evaluation of $T_M(x, y)$.

COMPUTING TUTTE POLYNOMIALS

Finite field method.

Let $\overline{\chi}(q,t) = (t-1)^r \, T\left(\frac{q+t-1}{t-1}, t\right)$.

Theorem. (A., 2002) Let $\mathcal{A}$ be a $\mathbb{Z}$-arrangement. Let q be a large prime, and let $\mathcal{A}_q$ be the induced arrangement in $\mathbb{F}_q^n$. Then

$$q^{n-r}\overline{\chi}_{\mathcal{A}}(q,t) = \sum_{p \in \mathbb{F}_q^n} t^{h(p)}$$

where $h(p)$ = number of hyperplanes of $\mathcal{A}_q$ that p lies on.

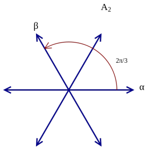
Computing Tutte polynomials is #P-hard, so we cannot expect miracles from this method. Still, it is often very useful.

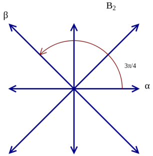
An application: Root systems.

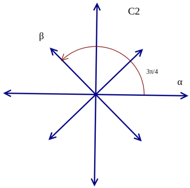
Root systems are arguably the most important vector configurations in mathematics. They are crucial in the classification of regular polytopes, simple Lie groups and Lie algebras, cluster algebras, etc.

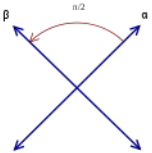
"Classical root systems":

$$
\begin{aligned}
A_n^+ &= \{e_i - e_j \,:\, 1 \leq i \leq j \leq n\} \\
B_n^+ &= \{e_i \pm e_j \,:\, 1 \leq i \leq j \leq n\} \cup \{e_i \,:\, 1 \leq i \leq n\} \\
C_n^+ &= \{e_i \pm e_j \,:\, 1 \leq i \leq j \leq n\} \cup \{2e_i \,:\, 1 \leq i \leq n\} \\
D_n^+ &= \{e_i \pm e_j \,:\, 1 \leq i \leq j \leq n\}
\end{aligned}
$$









TUTTE POLYNOMIALS OF CLASSICAL ROOT SYSTEMS

We can use this method to compute the Tutte polynomials of A_n, B_n, C_n, D_n. Surprisingly, they come from the **2-variable Rogers - Ramanujan function** from analytic number theory:

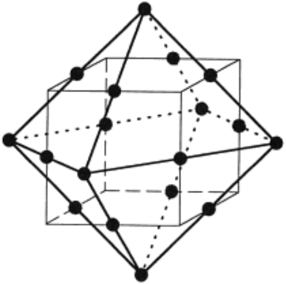
$$\sum_{n \geq 0} \frac{z^n y^{\binom{n}{2}}}{n!}$$

Theorem. [Tutte 67 / A. 02] The Tutte polynomials $\overline{\chi}_{A_n}(x, y)$ of the type A root systems are given by:

$$\left(\sum_{n \geq 0} \frac{z^n y^{\binom{n}{2}}}{n!} \right)^x = \overline{\chi}_{A_0}(x, y) \frac{z^0}{0!} + \overline{\chi}_{A_1}(x, y) \frac{z^1}{1!} + \overline{\chi}_{A_2}(x, y) \frac{z^2}{2!} + \cdots$$

Similar formulas hold for B_n, C_n, and D_n.
(More complicated, but also come from Rogers-Ramanujan function.)

muchas gracias



For more information, see:

http://math.sfsu.edu/federico
http://tinyurl.com/ardilamatroids