Coupled Elliptic Solvers for Embedded Mesh and Interface Problems

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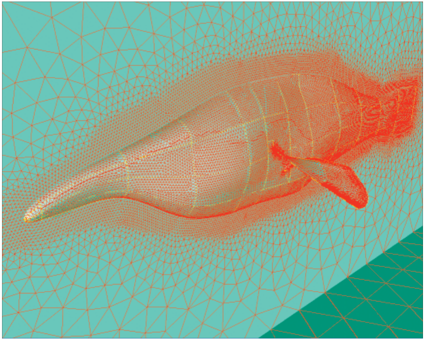
Complicated geometry, time-dependent problems...

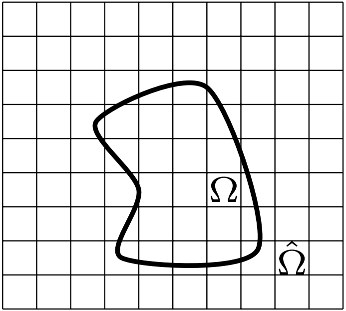
1. Conforming/refined mesh
   - Lose advantages of structured mesh
   - What if domain changes with time?



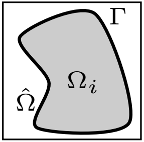
Source: Ramamurti et al., J Exp Biol. 205, 2997–3008 (2002)

Complicated geometry, time-dependent problems...

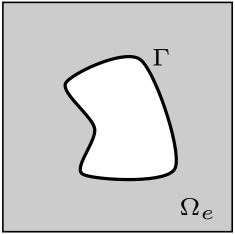
1. Conforming/refined mesh
   - Lose advantages of structured mesh
   - What if domain changes with time?
2. Embedded domain methods
   - How best to enforce the true boundary conditions?
   - Can we do this in a high-order way?



Embedded domain types



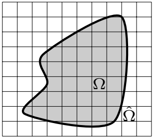
Interior problem on Ω_i embedded in a fictitious domain $\hat{\Omega}$



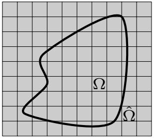
Problem in Ω_e is treated as a domain with an exclusion

Embedded domains & finite elements

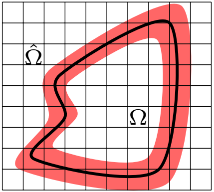
Finite cell methods
(e.g., Parvizian et al., 2007)



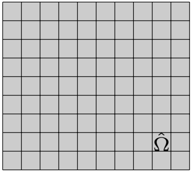
Fictitious domain methods
(e.g., Glowinski et al, 1994)



Immersed finite element methods



Finite element-integral equation method
(Rüberg & Cirak, 2010)



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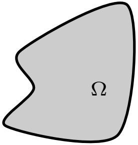


FE

Layer potential IE

Embedded mesh problems: convergence

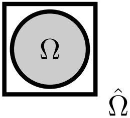
- High-order implementation of immersed boundary methods, immersed interface methods, etc. can be tricky
- Our FE-IE implementation **is high order when components are high order**

Embedded mesh problems: convergence

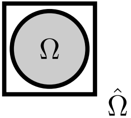
- Consider test problem:

$$\begin{aligned} -\nabla \cdot \nabla u(x) &= 1 \qquad x \in \Omega \\ u(x) &= 0 \qquad x \in \partial\Omega \end{aligned} \tag{1}$$

- Domain Ω is circle centered at $(0, 0)$ with radius $r = 0.5$; Embedded in $\hat{\Omega} = [-0.6, 0.6] \times [0.6, 0.6]$



Embedded mesh problems: convergence



- Splitting is:

$$
\begin{aligned}
\text{[FE]} \quad -\nabla \cdot \nabla u_1(x) &= 1 && x \in \hat{\Omega} \\
u_1 &= 0 && x \in \partial\hat{\Omega} \\
\text{[IE]} \quad -\nabla \cdot \nabla u_2(x) &= 0 && x \in \Omega \\
u_2 &= -u_1 && x \in \partial\Omega
\end{aligned} \tag{2}
$$

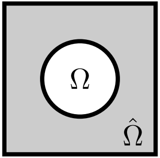
Embedded mesh problems: convergence

Convergence of our implementation for the interior FE-IE problem:

FE basis order	qbx	h_{fe}, h_{ie}	$\|\text{error}\|_\infty$	$\|\text{error}\|_0$	order
1	2	0.0300, 0.0786	2.32×10^{-4}	4.35×10^{-5}	–
		0.0150, 0.0393	5.07×10^{-5}	8.28×10^{-6}	2.39
		0.0075, 0.0196	9.35×10^{-6}	1.88×10^{-6}	2.14
2	3	0.0300, 0.0786	3.91×10^{-5}	9.16×10^{-6}	–
		0.0150, 0.0393	3.79×10^{-6}	8.93×10^{-7}	3.36
		0.0075, 0.0196	3.05×10^{-7}	7.03×10^{-8}	3.67
3	4	0.0300, 0.0786	8.02×10^{-6}	2.00×10^{-6}	–
		0.0150, 0.0393	4.33×10^{-7}	1.06×10^{-7}	4.24
		0.0075, 0.0196	1.81×10^{-8}	4.36×10^{-9}	4.60

Embedded mesh problems: convergence

Now consider the "complementary" problem
(domain with exclusion):



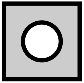
$$
\begin{aligned}
-\nabla \cdot \nabla u(x) &= f(x) \qquad && x \in \hat{\Omega} \backslash \Omega \\
u(x) &= g(x) && x \in \partial \Omega \\
u(x) &= \hat{g}(x) && x \in \partial \hat{\Omega}
\end{aligned} \tag{3}
$$

With $\hat{\Omega} = [-1, -1] \times [1, 1]$

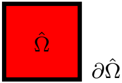
FE-IE for exclusion(s): new splitting

- Interior FE problem & purely exterior IE problem, coupled through boundary conditions → coupled system
- Split problems are now:

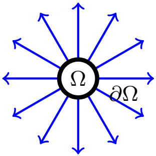
$$
\begin{aligned}
\text{[FE]} \quad & -\nabla \cdot \nabla u_1(x) = f(x) && x \in \hat{\Omega} \\
& u_1 = \hat{g}(x) - u_2 && x \in \partial\hat{\Omega} \\
\text{[IE]} \quad & -\nabla \cdot \nabla u_2(x) = 0 && x \in \mathbb{R}^2 \backslash \Omega \\
& u_2 = g(x) - u_1 && x \in \partial\Omega
\end{aligned} \tag{4}
$$



Original problem domain $\hat{\Omega}\backslash\Omega$



FE domain $\hat{\Omega} = [-1,-1] \times [1,1]$



IE domain $\mathbb{R}^2\backslash\Omega$

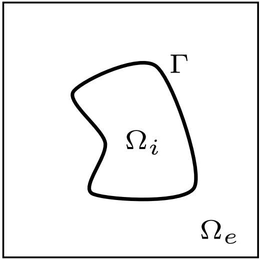
- **Achieve same order of convergence**

Coupled subproblems for interface problems

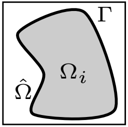
Consider the interface problem

$$
\begin{aligned}
-\beta \nabla \cdot \nabla u(x) &= f(x) \quad \text{in} \quad \Omega_i \cup \Omega_e, \quad \text{with} \\
u^i(x) &= c u^e(x) + a(x) \quad \text{on} \quad \Gamma, \quad \text{and} \\
\frac{\partial u^i(x)}{\partial n} &= \kappa \frac{\partial u^e(x)}{\partial n} + b(x) \quad \text{on} \quad \Gamma
\end{aligned} \tag{5}
$$

with two domains Ω_i and Ω_e separated by an interface Γ:



Coupled subproblems for interface problems



Interior problem on Ω_i embedded in a fictitious domain $\hat{\Omega}$

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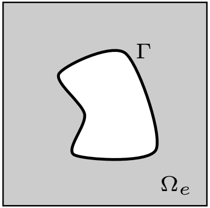
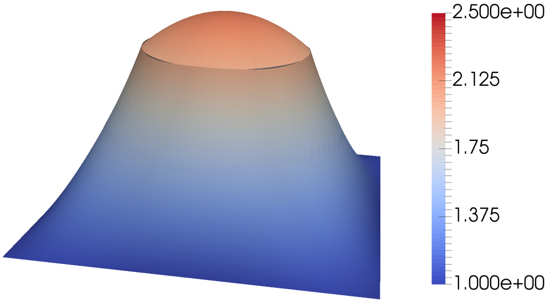


Problem in Ω_e is treated as a domain with an exclusion

$=$ coupled interior & exterior solutions

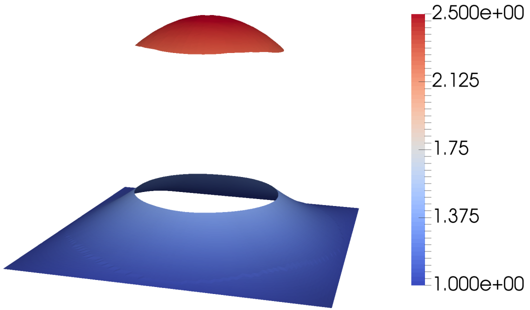
Coupled subproblems for interface problems

- Flexible representation through combinations of single & double layer potentials
- Can handle non-homogeneous jump conditions in derivative...



Coupled subproblems for interface problems

- Flexible representation through combinations of single & double layer potentials
- Can handle non-homogeneous jump conditions in derivative... and value



Interface problems: sample coupled system

$$\begin{bmatrix} \text{IE self op.} & \text{FE eval.} & \text{IE eval.} & \text{FE eval.} \\ & \text{FE matrix} & & \\ & \text{FE eval.} & \text{IE self op.} & \text{FE eval.} \\ \text{IE off-curve eval.} & & \text{IE off-curve eval.} & \text{FE matrix} \end{bmatrix} \begin{bmatrix} \sigma^i \\ U^i \\ \sigma^e \\ U^e \end{bmatrix} = \begin{bmatrix} \text{jump cond.} \\ \text{interior r.h.s.} \\ \text{jump cond.} \\ \text{exterior r.h.s} \end{bmatrix}$$

Summary

- Combine best aspects of FE and IE solvers
- Flexible representation for interior domains, domains with exclusions, and many interface problems
- Computational mechanics behind FE and IE solvers remain largely unchanged
- High-order convergence, even near the embedded boundary