A sketch of the statistical formalism for (emergent) phases

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1) *One defines a set X each point of which represents an infinite system of similar components.*
   Examples: A configuration of infinitely many unit cubes in $\mathbb{R}^3$; a simple graph on infinitely many nodes, ...
2) *There are projections of each point in X onto 'finite subsystems' labelled by v.*
   Examples: Those cubes intersecting the ball centered at the origin of $\mathbb{R}^3$ and radius v; the subgraph induced by the nodes in $v = \{1, 2, \dots, |v|\}, \dots$
3) *One chooses a few 'competing constraints' ϵ.*
   Examples: Cube configurations with given volume fraction and given density of 'energy of interaction'; graphs with given edge and triangle densities, ...
   The set of possible vector values of ϵ constitutes a constraint ('phase') space $\mathcal{Y}$, the boundary of which obviously corresponds to extrema of the constraints.
4) *One computes the number or volume $Z_v(\epsilon)$ of constrained projected configurations, and its exponential rate of growth $s(\epsilon) = \lim_{|v|\to\infty}(1/|v|)\log[Z_v(\epsilon)]$.*
   (I've cheated by leaving out the step of thickening the constraints by α and then taking $\alpha \to 0$ after the limit in v.)
5) *There is a simple real valued function R on a (relevant, 'probabilistic') space $\mathcal{F}_\epsilon$ such that $s(\epsilon) = R(\mu_\epsilon)$ for one or more 'equilibrium states' μ_ϵ in $\mathcal{F}_\epsilon$. The μ_ϵ are characterized alternatively by (local) DLR equations, or as maximizers of $R(\mu)$ among all μ in $\mathcal{F}_\epsilon$.*
   (The μ_ϵ give a much richer picture of the system than the function $s(\epsilon)$.)

Notes on the above

i) Both characterizations in 5) of equilibrium states μ_ϵ were proven, for stat mech 'on lattices', in the 1960's. However for more realistic models, such as particles or cubes in a continuum such as $\mathbb{R}^3$, the DLR characterization of μ_e was proven by Dobrushin and Lanford/Ruelle around 1969 but I don't believe the variational characterization was ever proven for such systems. For graphs the opposite is true: the variational characterization is proven but I don't think there is any analogue of the DLR characterization of the entropy maximizers μ_ϵ.

ii) A simplified version of the Gibbs phase rule, unproven except in specific examples, states: 'Except for a set of constraints ϵ of lower dimension, $s(\epsilon)$ is real analytic at ϵ.' The main point of the formalism: There exist simply stated examples (e.g. 'hard spheres') with multiple phases, i.e. for which the region $\mathcal{A} \subset \mathcal{Y}$ of analyticity of $s(\epsilon)$ is not connected.

iii) The formalism 1) – 5) seems to be easier to develop/analyze in graphs than in stat mech. In particular the μ_ϵ are simple enough to 'understand' (multipodal) in graph models.

iv) Note that for stat mech one can start in the grand canonical ensemble and derive the characterizations of 5) using the Legendre transform, but this does not work for graphs; the grand canonical ensemble loses information, and is misleading, for graphs.

v) A large deviations principle can be useful to derive the variational characterization of 5) but is not necessary; the variational characterization was first developed in lattice stat mech by Ruelle (1965, 1967) without an underlying large deviations principle, though this path was later derived in the 1980's.

Some questions

a) Do we want to say that the system of noninteracting particles forms an emergent 'phase'?

b) Why can material phases be determined by so few parameters?

D. Ruelle: J. Math. Phys. 6 (1965) 201-220; Commun. math. Phys. 5 (1967) 324-329

O.E. Lanford and D. Ruelle: Commun. math. Phys. 13 (1969) 194-215