Borcherds algebras and finite order automorphisms

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Outline

DEFINITION 1

Borcherds' theorem, which can be used as a definition.
Let $\mathfrak{g}$ be a Lie algebra satisfying the following conditions:

- $\mathfrak{g}$ can be $\mathbb{Z}$-graded as $\coprod_{i\in\mathbb{Z}} \mathfrak{g}_i$, $\mathfrak{g}_i$ is f.d. if $i \neq 0$, and $\mathfrak{g}$ is diagonalizable w.r.t $\mathfrak{g}_0$.
- $\mathfrak{g}$ has an involution ω which maps $\mathfrak{g}_i$ onto $\mathfrak{g}_{-i}$
- $\mathfrak{g}$ has an invariant bilinear form $(\cdot,\cdot)$, invariant under ω, such that $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal if $i \neq -j$, and such that the form $(\cdot,\cdot)_0$, defined by $(x,y)_0 = -(x,\omega(y))$ for $x, y \in \mathfrak{g}$, is positive definite on $\mathfrak{g}_m$ if $m \neq 0$.
- $\mathfrak{g}_0 \subset [\mathfrak{g},\mathfrak{g}]$.

Then we call $\mathfrak{g}$, and any central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ a generalized Kac-Moody (or Borcherds) algebra.

MOTIVATING EXAMPLE

Some interesting examples of generalized Kac-Moody algebras are those constructed using vertex algebras. Such examples include the “monster" Lie algebra and “fake monster" Lie algebra.
For the monster case, let M denote the monster simple group. Then consider the special graded vector space $V^\natural$ (graded by “energy" i.e. conformal weight). The v.s. $V^\natural$ is the “moonshine module" of M, and $M = Aut(V^\natural)$ (voa).

As a vector space, and M-module

$$V^\natural = \coprod_{i \geq 0} V^\natural_{(i)}.$$

Here, each homogenous space $V^\natural_{(i)}$ is a finite dimensional M-module.

The subspace $V^\natural_{(0)}$ is the trivial module, $V^\natural_{(1)} = \emptyset$, $V^\natural_{(2)}$ a 196884-dimensional module (isomorphic to the Greiss algebra), etc.
If we shift the grading so that

$$V^\natural = \coprod_{i \geq -1} V^\natural_i.$$

then the generating function for the dimensions is given by the modular function $j(q) - 744 = c(-1)q^{-1} + c(1)q + c(2)q^2 + \cdots$. [FLM]
This vector space $V^\natural$ can be given the structure of a vertex operator algebra. [Borcherds, FLM]
From here you can read about Monstrous Moonshine :)

Construction of the Lie Algebra from the Moonshine module.

A vertex algebra V is a vector space equipped with (noncommutative, nonassociative) multiplications

$$u \cdot_n v$$

for $u, v \in V$, $n \in \mathbf{Z}$. (These multiplications satisfy various identities and axioms.)
Equivalently, for each $n \in \mathbf{Z}$ and $u \in V$ we have a map $u_n \in \mathrm{End}(V)$ and we write

$$u \cdot_n v = u_n(v) = u_n v$$

The functions u_n are called vertex operators. In this lecture we are mostly interested in $n = 0$.

Given V a vertex algebra with conformal vector, and therefore an action of the Virasoro algebra $\{L_n : n \in \mathbb{Z}\}$, let

$$P_i = \{v \in V | L(0)v = iv, L(n)v = 0 \text{ if } n > 0\}.$$

(P_1 is called the physical space).
The space $P_1/L(-1)P_0$ is a Lie algebra with Lie product given by

$$[u + L(-1)P_0, v + L(-1)P_0] = u_0 v + L(-1)P_0.$$

Let $V = V^\natural \otimes V_{1,1}$ where $V_{1,1}$ is a 2-dimensional Lorentzian lattice vertex algebra. The monster Lie algebra is

$$\mathfrak{m} = P_1/L(-1)P_0$$

for this V.

That $\mathfrak{m}$ satisfies the conditions of Definition 1 follows from the properties of vertex algebras.

The (specialized) denominator identity for $\mathfrak{m}$ is

$$p(j(p) - j(q)) = \prod_{\substack{i \in \mathbb{Z}_+ \\ j \in \mathbb{Z}_+ \cup \{-1\}}} (1 - p^i q^j)^{c(ij)} \tag{1}$$

The $c(ij)$ are the dimensions of the $(i,j)^{th}$-homogenous subspace of $\mathfrak{m}$.

Generalization of a Cartan matrix

Let I be a (finite or) countable index set and let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{C}$, satisfying the following conditions:

(B1) A is symmetric.

(B2) If $i \neq j$ $(i, j \in I)$, then $a_{ij} \leq 0$.

(B3) If $a_{ii} > 0$ $(i \in I)$, then $2a_{ij}/a_{ii} \in \mathbb{Z}$ for all $j \in I$.

Call such a matrix a GGC or Borcherds matrix.

THE MONSTER EXAMPLE

$$A = \begin{array}{c} \\ c(1)\updownarrow \\ \\ c(2)\updownarrow \\ \\ \end{array} \left(\begin{array}{c|ccc|ccc|c} & \multicolumn{3}{c|}{\overleftrightarrow{\quad c(1)\quad}} & \multicolumn{3}{c|}{\overleftrightarrow{\quad c(2)\quad}} & \\ 2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\ \hline 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ 0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \hline -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\ -1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \hline \vdots & & \vdots & & & \vdots & & \end{array}\right)$$

The terms $c(i)$ are coefficients of q for the modular function $j(q) - 744$

Note that this matrix is infinite, and highly degenerate. In the monster (and baby monster) example, the size of the blocks are determined by coefficients of modular functions. The negative diagonal entries correspond to norms of “imaginary" simple roots.

GENERATORS AND RELATIONS

Let $\mathfrak{g}'(A)$ be the Lie algebra with generators h_i, e_i, f_i, $i \in I$, and the following defining relations: For all $i, j, k \in I$,

$$[h_i, h_j] = 0, [e_i, f_j] - \delta_{ij} h_i = 0,$$

$$[h_i, e_k] - a_{ik} e_k = 0, [h_i, f_k] + a_{ik} f_k = 0$$

and Serre relations

$$(\mathrm{ad}\, \mathrm{e_i})^{-2\mathrm{a_{ij}}/\mathrm{a_{ii}}+1} \mathrm{e_j} = 0, (\mathrm{ad}\, \mathrm{f_i})^{-2\mathrm{a_{ij}}/\mathrm{a_{ii}}+1} \mathrm{f_j} = 0$$

for all $i \neq j$ with $a_{ii} > 0$, and finally

$$[e_i, e_j] = 0, [f_i, f_j] = 0$$

whenever $a_{ij} = 0$.

We extend the Lie algebra by an appropriate abelian Lie algebra $\mathfrak{d}$ of degree derivations, chosen so that the simple roots are linearly independent in $(\mathfrak{h} \ltimes \mathfrak{d})^*$.
The Lie algebra $\mathfrak{g}(A) = g'(A) \ltimes \mathfrak{d}$ is the Borcherds or generalized Kac-Moody (Lie) algebra associated to the matrix A.
The monster Lie algebra $\mathfrak{m}$ is isomorphic to $\mathfrak{g}(A)/\mathfrak{c}$ where $\mathfrak{c}$ the full center of $\mathfrak{g}(A)$ for the above A

The monster Lie algebra can be graded by $\mathbf{Z} \times \mathbf{Z}$, and we now collect the generators into their block form:

Grading

- e_{jk} has degree $(1, j), j = -1, 1, 2, \cdots, 1 \leq k \leq c(j)$
- f_{jk} has degree $-(1, j), j = -1, 1, 2, \cdots, 1 \leq k \leq c(j)$
- h_{jk} has degree $(0, 0)$ $j = -1, 2, \cdots, 1 \leq k \leq c(j)$

This $\mathbf{Z} \times \mathbf{Z}$- grading yields the specialization of the denominator identity of $\mathfrak{g}(A)$ that is the product formula for the modular function $j(\tau)$.

FREE SUBALGEBRAS

Both the monster and baby monster Lie algebra satisfy the conditions of the theorem of [Ju 1998].

Theorem

Let $\mathfrak{g}(A) = \mathfrak{g}$ be a generalized Kac-Moody algebra such that if α_i and α_j are two distinct imaginary simple roots then $a_{ij} < 0$. Let $S = \{i \in I | a_{ii} > 0\}$. Then $\mathfrak{g} = \mathfrak{u}^+ \oplus \mathfrak{r} \oplus \mathfrak{u}^-$, where $\mathfrak{u}^-$ is the free Lie algebra on the direct sum of the integrable highest weight $\mathfrak{g}_S$-modules $U(\mathfrak{n}_S^-) \cdot f_j$ for $j \in I \backslash S$ and $\mathfrak{u}^+$ is the free Lie algebra on the direct sum of the integrable lowest weight $\mathfrak{g}_S$-modules $U(\mathfrak{n}_S^+) \cdot e_j$ for $j \in I \backslash S$.

In particular for both the monster and baby monster, there is only one simple real root. Thus in the decomposition

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{r} \oplus \mathfrak{u}^+$$

we have $\mathfrak{g}_S \simeq \mathfrak{sl}_2$, and $\mathfrak{r} = \mathfrak{gl}_2$. The $\mathfrak{u}^\pm$ are then generated as free Lie algebras by the appropriate lowest (resp. highest) $\mathfrak{sl}_2$ weight modules.

SET OF FREE GENERATORS OF $\mathfrak{u}^{\pm}$

In the case of the monster Lie algebra, the free Lie subalgebra $\mathfrak{u}^+$ is generated by the set

Free generators in $\mathfrak{u}^+ \subset \mathfrak{m}$

$U = \cup_{j \in \mathbf{N}} \{ (ad\, e_{-1})^l e_{jk} \mid 0 \leq l < j, 1 \leq k \leq c(j) \}$

Here, the $c(j)$ are the coefficients of the modular function $J(q) = j(q) - 744$, (for the baby monster use a different modular fn $F(q^2)$)

The product formula and denominator identity for $\mathfrak{m}$ is completely determined by the free part, and $\mathfrak{sl}_2$.
Using a mild generalization of a product formula appearing in Bourbaki for the dimensions of a free Lie algebra, because our algebra $\mathfrak{u}^+$ is graded by a lattice:

$$1 - \sum_{(i,j)\in\mathbb{N}^2\setminus\{0\}} c(i+j-1)u^i v^j = \prod_{(i,j)\in\mathbb{N}^2\setminus\{0\}} (1-u^i v^j)^{\dim u^{(i,j)}}. \quad (2)$$

(it's worth noting that all one needs to reproduce the computational aspect of B's proof of monstrous moonshine one can work the free subalgebra, and can avoid actually having to generalize K-M theory at all–)

In particular, once one has the product formula for the free Lie subalgebra, solve for $c(ij)$ using mobius inversion (we can do this for all the graded traces of elements of M, not jut the idenitity)
In order to obtain recursion relations for the McKay-Thompson series, we apply Möbius inversion to (2),

$$c_g(ij) = \sum_{\substack{k>0 \\ k(m,n)=(i,j)}} \frac{1}{k}\mu(k)\Psi^k \left(\sum_{a\in P(m,n)} \frac{(|a|-1)!}{a!} \prod_{r,s\in\mathbb{Z}_+} c_g(r+s-1)^{a_{rs}} \right)$$

$$= \sum_{\substack{k>0 \\ k(m,n)=(i,j)}} \frac{1}{k}\mu(k) \left(\sum_{a\in P(m,n)} \frac{(|a|-1)!}{a!} \prod_{r,s\in\mathbb{Z}_+} c_{g^k}(r+s-1)^{a_{rs}} \right). \tag{3}$$

Note that setting $g=e$ in equation (3) yields the multiplicity formula generalized in [Kang]. (Ψ^k is the Adams operator)

To obtain the specialization of the denominator formula of $\mathfrak{g}$ we must include the degree $(1,-1)$ subspace $\mathfrak{g}^{(1,-1)} = \mathbb{R}e_{-1}$, which is one-dimensional. We multiply both sides of the product formula by $(1-u/v)$ (recall $e_{-1} \to (1,-1)$)

$$\prod_{(i,j)} (1-u^i v^j)^{\dim \mathfrak{g}^{(i,j)}} = \prod_{(i,j)\in\mathbb{N}^2-\{0\}} (1-u^i v^j)^{\dim L^{(i,j)}} (1-u/v)$$

Now we can do some algebra and recover the product formula

$$= (1 - \sum_{(i,j) \in \mathbb{N}^2 - \{0\}} c(i+j-1) u^i v^j)(1 - u/v)$$

$$= 1 - \sum c(i+j-1) u^i v^j - u/v + \sum c(i+j-1) u^{i+1} v^{j-1}$$

$$= u(J(u) - J(v)).$$

There is a product formula for the modular function j (see Borcherds) which can be written:

$$p(j(p) - j(q)) = \prod_{\substack{i=1,2,\dots \\ j=-1,1,\dots}} (1 - p^i q^j)^{c(ij)}, \qquad (4)$$

which converges on an open set in $\mathbb{C}$, and so implies the corresponding identity for formal power series. Now we conclude that $\dim \mathfrak{g}^{(i,j)} = c(ij)$. $\square$

Problem for SAGE program:

Elements of M act on $\mathfrak{m}$, where the action is inherited from the moonshine module $V^\natural$. In particular,

1. M acts trivially on the $\mathfrak{gl}_2$ component
2. The action of M respects the $Z \times Z$ lattice grading
3. The fixed subalgebras $\mathfrak{u}^{\pm}(\sigma) = \{a \in u^+ \mid \sigma(a) = a\}$ is also a free Lie algebra

So that the fixed point algebra $\mathfrak{m}^\sigma$ is GKM of a similar type as the monster: It's Z^2-graded, and an M-module, with denominator identity determined by the $\mathfrak{sl}_2$ and free component.

$$\mathfrak{m}^\sigma = \mathfrak{u}^-(\sigma) \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}^+(\sigma)$$

QUESTIONS:
1) Can we count generators of degree (i,j), thereby solving recursions for the dimensions of the homogenous components of the algebre $\mathfrak{m}^{\sigma}$?
2) What is a good set of generators for the fixed subalgebra $\mathfrak{u}^{+}(\sigma)$? One that leads to a nice product formula for the denominator identity?
3) If we choose Hohn's order 2 automorphism how far off are we from his generalized moonshine result?
NEXT: Act on the new algebra by a (commuting) element $\nu \in M$ and compute the graded trace, graded characters etc.

FYI Graded dimension of a Free Lie algebra

Serre's formula for the rank (or dimension) of the homogeneous subspaces of the free Lie algebra $L(X) = \sum_\alpha L^\alpha(X)$ (or the free associative algebra) generated by a set $X = \{x_i | i \in I\}$:

$$1 - \sum_{i \in I} T_i = \prod_{\alpha \in \Delta_{|X|} \setminus \{0\}} (1 - T^\alpha)^{c(\alpha)}$$

Which has a generalization

$$1 - \sum_{\alpha \in \Delta} n_\alpha T^\alpha = \prod_{\alpha \in \Delta \setminus \{0\}} (1 - T^\alpha)^{c(\alpha)}.$$

Here there are n_α generators of the free Lie algebra of degree α, and $c(\alpha)$ is the rank of $L^\alpha(X)$.

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