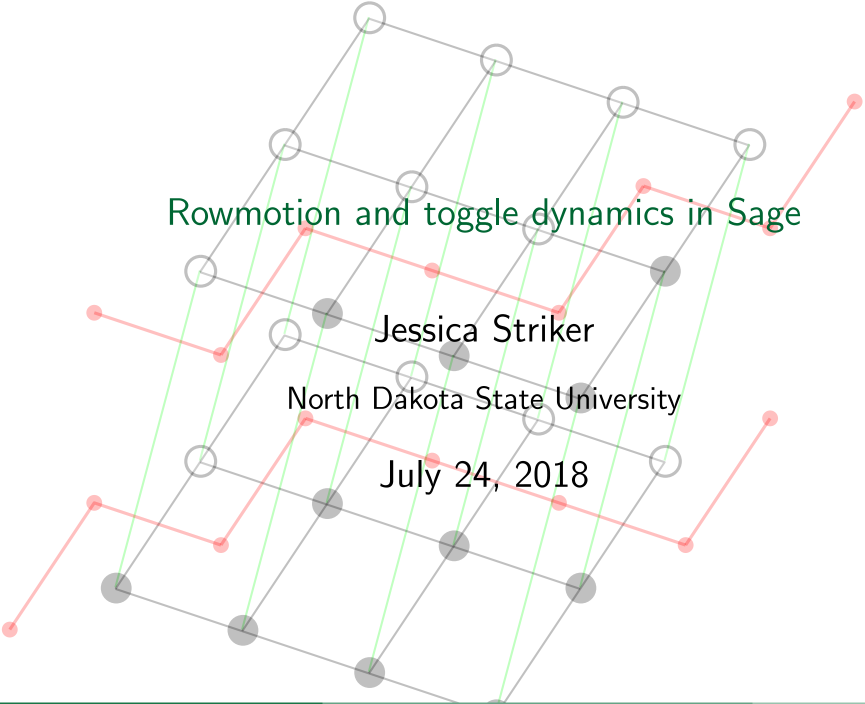

Rowmotion and toggle dynamics in Sage

Jessica Striker

North Dakota State University

July 24, 2018

Rowmotion and toggle dynamics in Sage

1. Promotion and rowmotion
   - Posets and order ideals
   - Rowmotion
   - Toggles
2. Cyclic sieving
3. Homomesy
4. Multidimensional promotion and rowmotion
5. Increasing labeling promotion and rowmotion

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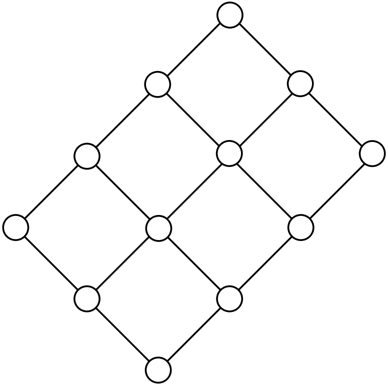
Start with a poset

A **poset** is a **p**artially **o**rdered **set**.

Definition

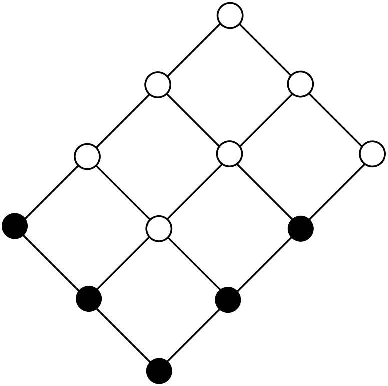
A *poset* is a set with a partial order " $\leq$ " that is reflexive, antisymmetric, and transitive.



Specify a set of subsets (order ideals)

Definition

An *order ideal* of a poset P is a subset $X \subseteq P$ such that if $y \in X$ and $z \leq y$, then $z \in X$. The set of order ideals of P is denoted $J(P)$.



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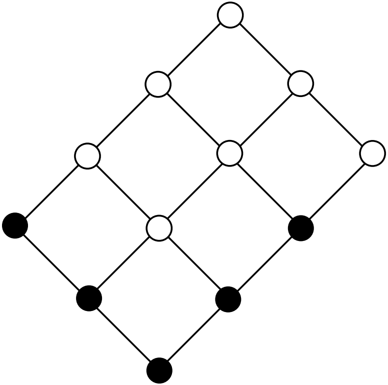
Rowmotion

Definition

The **rowmotion** of an order ideal X is the order ideal generated by the minimal elements of P not in X.

An order ideal X

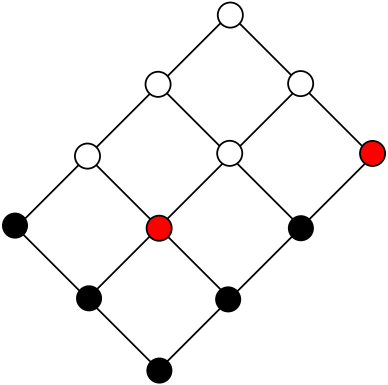


Rowmotion

Definition

The **rowmotion** of an order ideal X is the order ideal generated by the minimal elements of P not in X.

Find the **minimal** elements of P not in X.

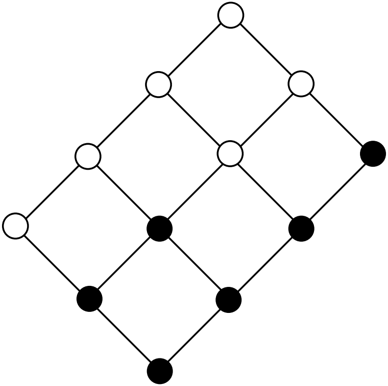


Rowmotion

Definition

The **rowmotion** of an order ideal X is the order ideal generated by the minimal elements of P not in X.

Use them to generate a new order ideal **Row(X)**.

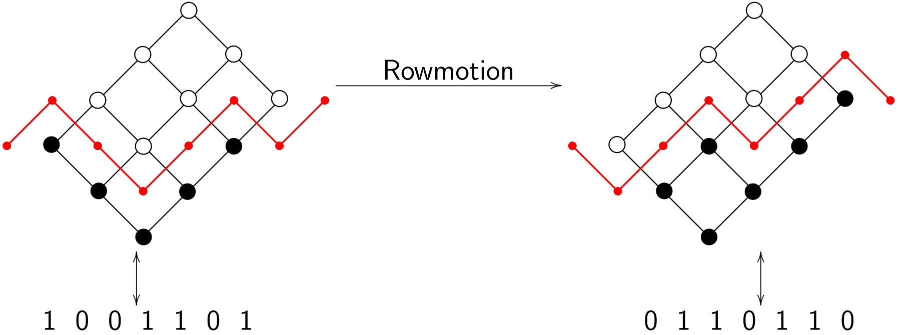


Rowmotion in Sage

Rowmotion in rectangular posets $\mathbf{a} \times \mathbf{b}$

Theorem (A. Brouwer and A. Schrijver 1974)

The order of rowmotion on $J(\mathbf{a} \times \mathbf{b})$ is $a + b$.



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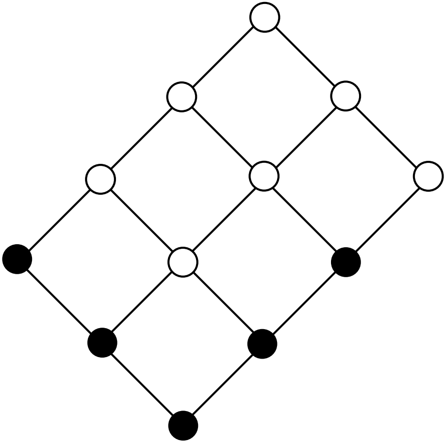
What is a toggle?

www.amazon.com/Shoreline-Marine-Toggle-Switch-Brass/dp/B004LR5N3C

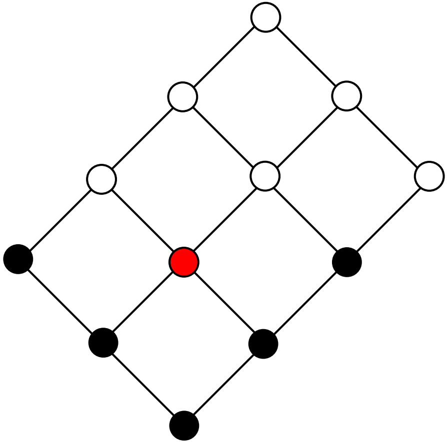
Toggles act on order ideals

Define a toggle, t_e, for each $e \in P$.



Toggles act on order ideals

Toggles t_e add e when possible.



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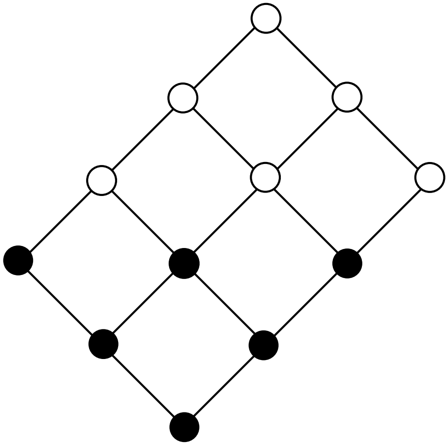

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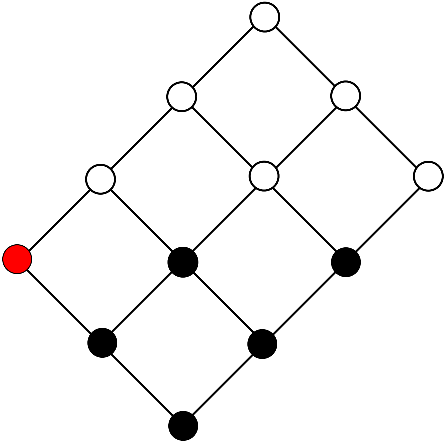

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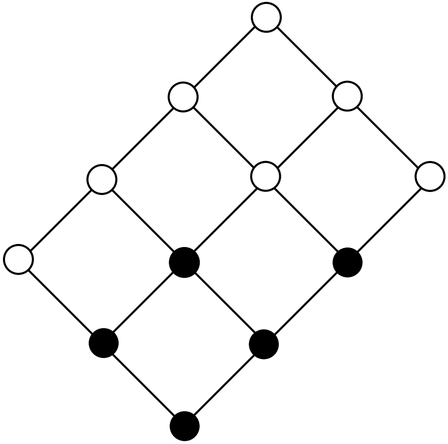
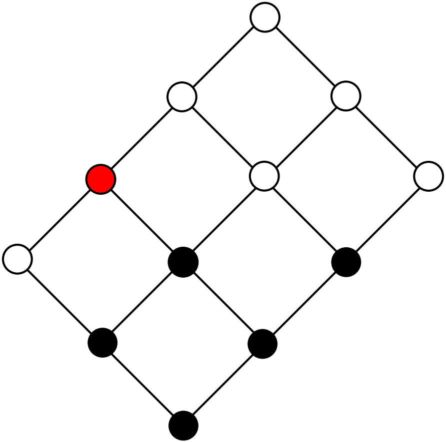


Toggles act on order ideals

Toggles t_e do nothing otherwise.



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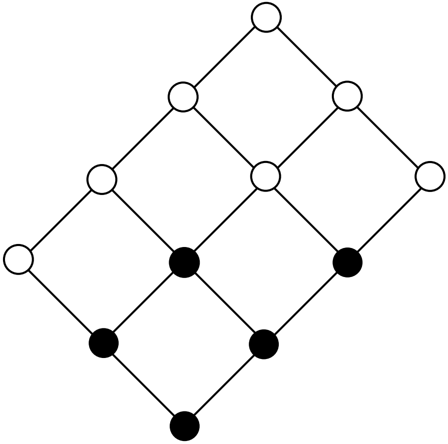


Order ideal toggles

Let P be a poset and $J(P) \subseteq 2^P$ its set of order ideals.

Definition

For each element $e \in P$ define its *toggle* $t_e : J(P) \to J(P)$ as:

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in J(P) \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in J(P) \\ X & \text{otherwise.} \end{cases}$$

Generalized toggles

Let E be a finite set and $\mathcal{L} \subseteq 2^E$ be any set of subsets.

Definition (S. 2018)

For each element $e \in E$ define its *toggle* $t_e : \mathcal{L} \to \mathcal{L}$ as:

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in \mathcal{L} \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in \mathcal{L} \\ X & \text{otherwise.} \end{cases}$$

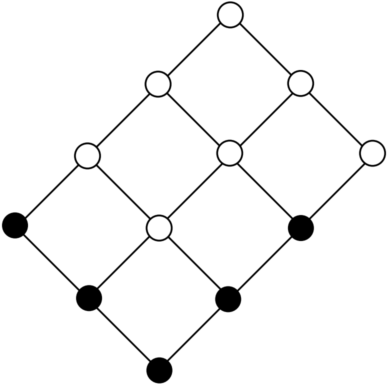
Generalized toggles have been studied in the following settings:

- antichains of a poset (M. Joseph 2018+)
- independent sets of a graph (M. Joseph and T. Roby 2017)
- noncrossing partitions (D. Einstein, M. Farber, E. Gunawan, M. Joseph, M. Macauley, J. Propp, S. Rubinstein-Salzedo 2016)

Rowmotion on order ideals is a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let P be a poset and X an order ideal of P. Then rowmotion on X is the order ideal obtained from X by toggling the elements of P from top to bottom.



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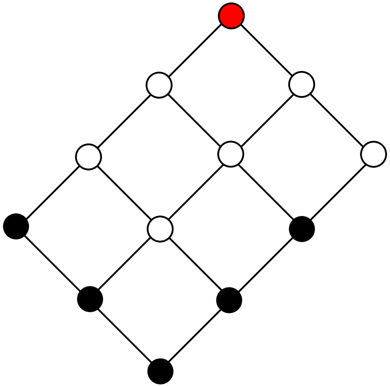

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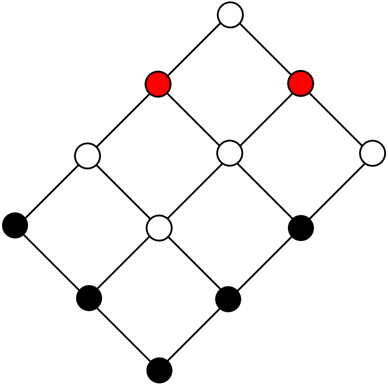

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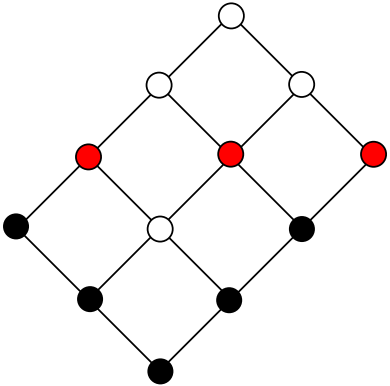

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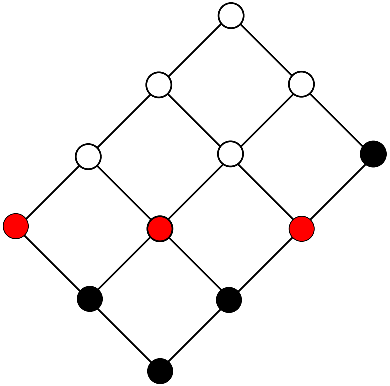

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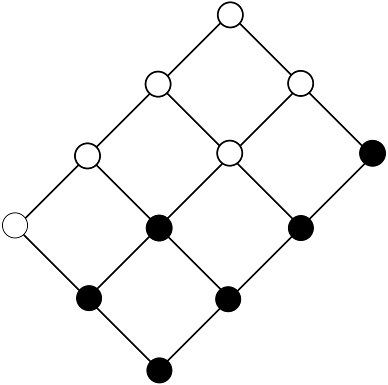

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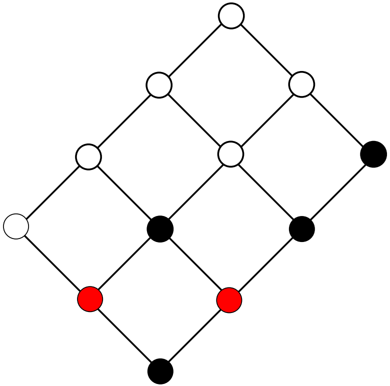

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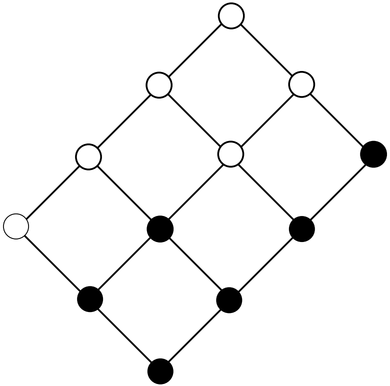

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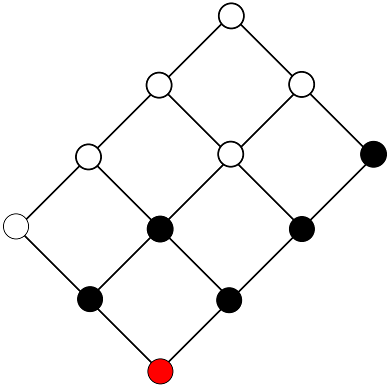

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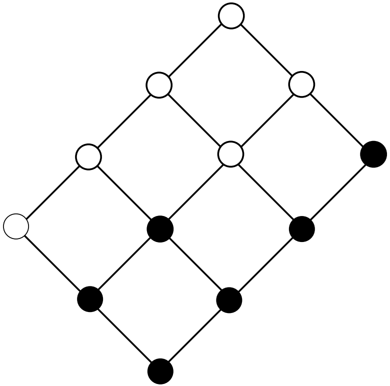
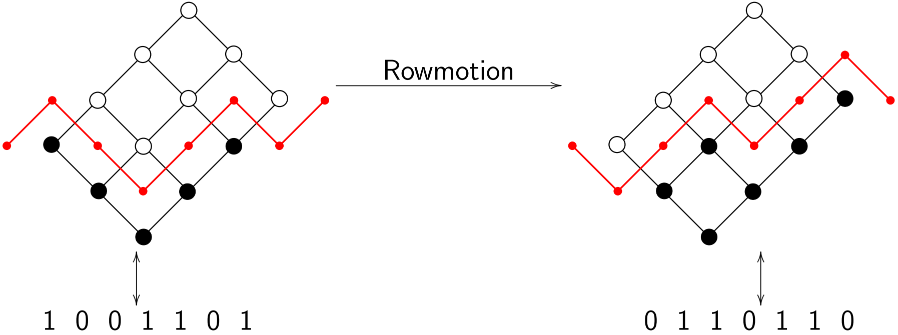


Rowmotion in rectangular posets $\mathbf{a} \times \mathbf{b}$

Theorem (A. Brouwer and A. Schrijver 1974)

The order of rowmotion on $J(\mathbf{a} \times \mathbf{b})$ is $a + b$.



Promotion and rowmotion are conjugate actions

Theorem (N. Williams and S. 2012)

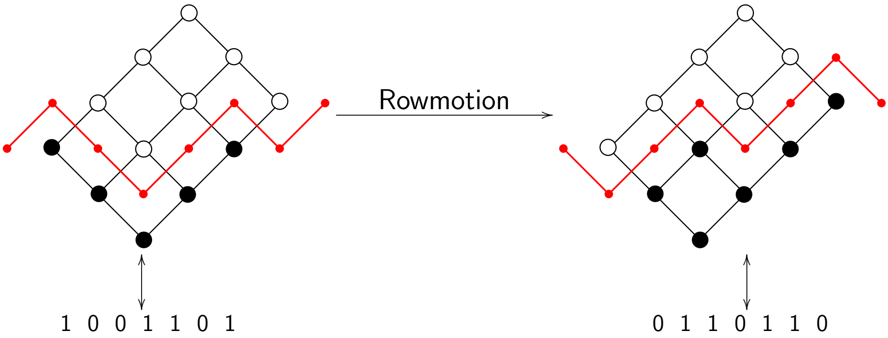
In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

Promotion and rowmotion
have the same orbit structure!

Rowmotion in rectangular posets $\mathbf{a} \times \mathbf{b}$

Corollary (N. Williams and S. 2012)

There is an equivariant bijection between order ideals of $\mathbf{a} \times \mathbf{b}$ under rowmotion and binary words of length $a + b$ with b ones under a cyclic shift. So rowmotion has order $a + b$.



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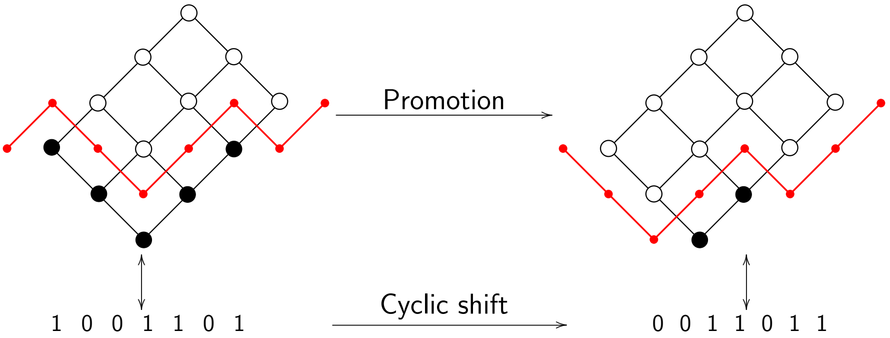


Rowmotion in triangular posets $\mathbf{A}_b$ ($b-1$ min elements)

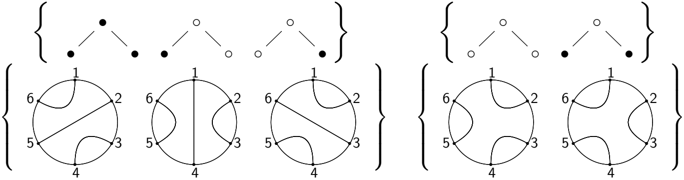
Theorem (D. Armstrong, C. Stump, H. Thomas 2013)

The order of rowmotion on $J(\mathbf{A}_b)$ is $2b$.

Explanation (as a corollary of the theorem on a previous slide):

Corollary (N. Williams and S. 2012)

There is an equivariant bijection between the order ideals of $\mathbf{A}_b$ under rowmotion and noncrossing matchings of $2b$ under rotation. So rowmotion has order $2b$.



$$\left\{\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline\end{array}\right\} \qquad \left\{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array}\right\}$$

Rowmotion in $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$

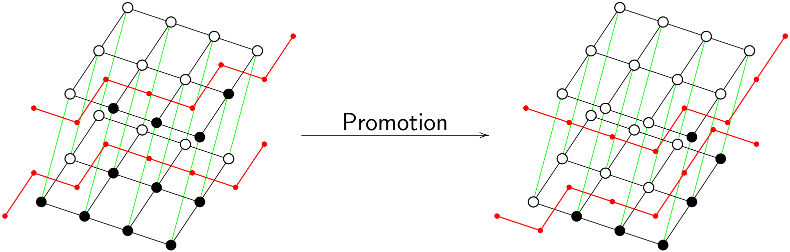
Theorem (P. Cameron and D. Fon-der-Flaass 1995)

The order of rowmotion on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ is $a + b + 1$.

Explanation:

Theorem (N. Williams and S. 2012)

There is an equivariant bijection between order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ under rowmotion and noncrossing partitions of $a + b + 1$ into $b + 1$ blocks under rotation. So rowmotion has order $a + b + 1$.



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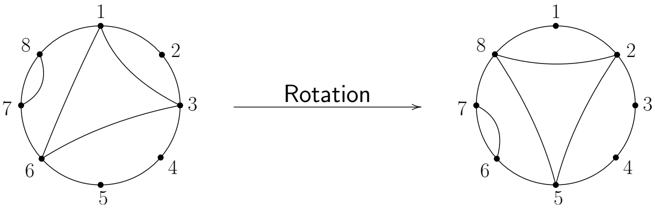
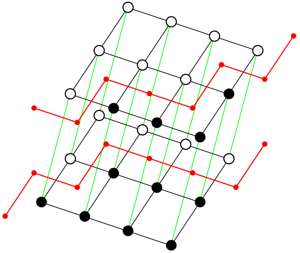
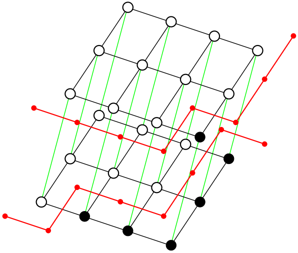


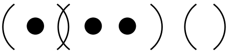


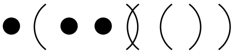
Promotion $\longrightarrow$

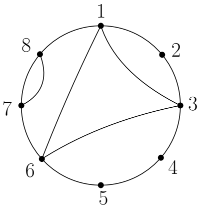


1	0	1	0	0	0	1	0
0	0	1	0	0	1	0	1

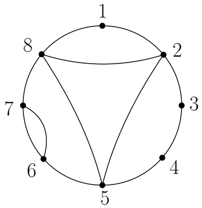
0	1	0	0	1	1	0	0
0	0	0	0	1	0	1	1







Rotation $\longrightarrow$



What is the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$?

- The order of rowmotion on $\mathbf{a} \times \mathbf{b}$ is $a + b$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ is $a + b + 1$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{3}$ is $a + b + 2$ for as high as we can check.
- This may lead one to conjecture that the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ is $a + b + c - 1$.

What is the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$?

- The order of rowmotion on $\mathbf{a} \times \mathbf{b}$ is $a + b$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ is $a + b + 1$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{3}$ is $a + b + 2$ for as high as we can check.
- This may lead one to conjecture that the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ is $a + b + c - 1$.
- But the orbits of rowmotion on $\mathbf{4} \times \mathbf{4} \times \mathbf{4}$ are of size 11 and 33. (We'll come back to this later in the talk.)

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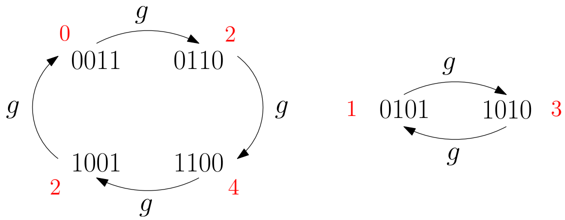
The cyclic sieving phenomenon

Definition (V. Reiner, D. Stanton, D. White 2004)

Given a set S, a polynomial $f(q)$, and a bijective action g of order n, the triple $(S, f(q), g)$ exhibits the *cyclic sieving phenomenon* if $f(\zeta^d)$, where $\zeta = e^{2\pi i/n}$, counts the elements of S fixed under g^d.



$f(q) = \sum_{w \in \binom{4}{2}} q^{\mathrm{inv}(w)} = 1 + q + 2q^2 + q^3 + q^4, \qquad \zeta = e^{2\pi i/4} = i$

$f(i^1) = 1 + i + -2 - i + 1 = 0$, and 0 elements are fixed under g^1.

$f(i^2) = f(-1) = 2$, and 2 elements are fixed under g^2.

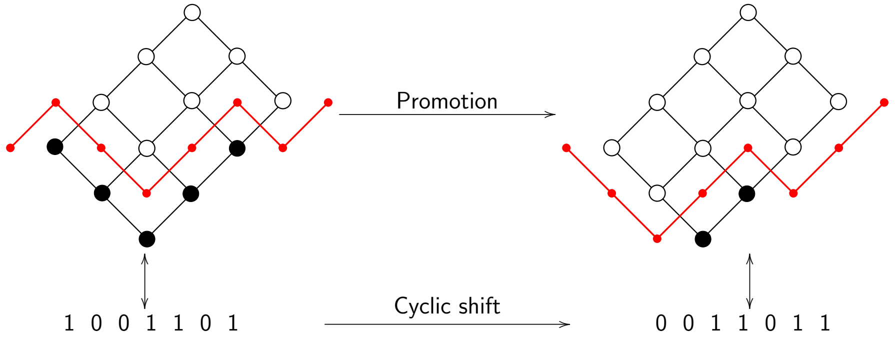
$f(i^3) = f(-i) = 0$, and 0 elements are fixed under g^3.

$f(i^4) = f(1) = 6$, and 6 elements are fixed under g^4.

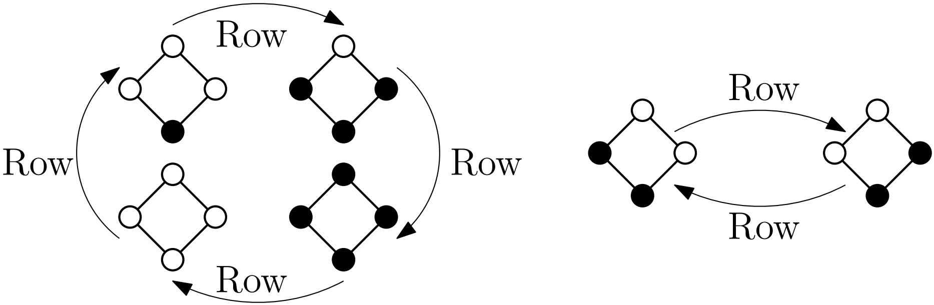
Cyclic sieving of rowmotion on $\mathbf{a} \times \mathbf{b}$

Corollary (N. Williams and S. 2012)

There is an equivariant bijection between the order ideals of $\mathbf{a} \times \mathbf{b}$ under rowmotion and binary words of length $a + b$ with b ones under rotation. So $(J(\mathbf{a} \times \mathbf{b}), f, \mathrm{Row})$ exhibits the cyclic sieving phenomenon, where $f(q) = \sum_{w \in \binom{4}{2}} q^{\mathrm{inv}(w)} = \sum_{X \in J(P)} q^{|X|}$.



Cyclic sieving of rowmotion on $\mathbf{2} \times \mathbf{2}$



$f(q) = 1 + q + 2q^2 + q^3 + q^4 \qquad \zeta = e^{2\pi i/4} = i$

$f(i^1) = 0$, so 0 elements are fixed under Row^1

$f(i^2) = f(-1) = 2$, so 2 elements are fixed under Row^2

$f(i^3) = f(-i) = 0$, so 0 elements are fixed under Row^3

$f(i^4) = f(1) = 6$, so 6 elements are fixed under Row^4

Cyclic sieving of promotion on $SYT(2 \times b)$

Theorem (Haiman 1992)

Every $a \times b$ standard Young tableau is fixed by promotion to the power ab.

Theorem (Rhoades 2010)

$(SYT(a \times b), f, \mathrm{Pro})$ *exhibits the cyclic sieving phenomenon, where f is the q-analogue of the hook-length formula.*

Corollary (Rhoades 2010)

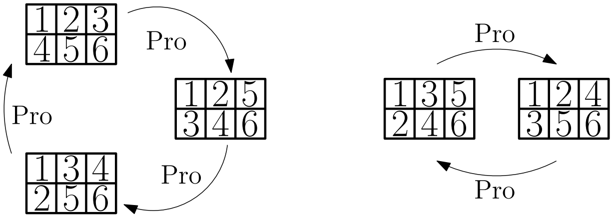
$(SYT(2 \times b), f, \mathrm{Pro})$ *exhibits the cyclic sieving phenomenon, where* $f(q) = \dfrac{(2n)!_q}{(n+1)!_q n!_q}$.

Note $[n]_q = (1 + q + q^2 + \cdots + q^{n-1})$ and $n!_q = [n]_q[n-1]_q \cdots [2]_q$.

Cyclic sieving of promotion on $SYT(a \times b)$

Corollary (Rhoades 2010)

$(SYT(2 \times b), f, \mathrm{Pro})$ *exhibits the CSP, where* $f(q) = \dfrac{(2n)!_q}{(n+1)!_q n!_q}$.



$f(q) = \frac{6!_q}{4!_q 3!_q} = 1 + q^2 + q^3 + q^4 + q^6$ $\qquad \zeta = e^{2\pi i/6} = e^{\pi i/3}$

$f(\zeta^1) = f(\zeta^4) = f(\zeta^5) = 0$, so 0 elements fixed under Pro^1, Pro^4, Pro^5.

$f(\zeta^2) = 2$, so 2 elements are fixed under Pro^2.

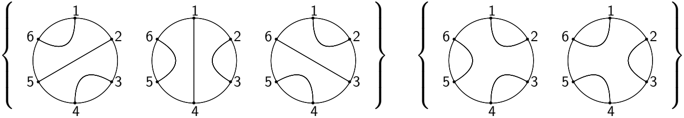
$f(\zeta^3) = f(-1) = 3$, so 3 elements are fixed under Pro^3.

$f(\zeta^6) = f(1) = 5$, so 5 elements are fixed under Pro^6.

Cyclic sieving of rowmotion in triangular posets $\mathbf{A}_b$

Corollary (N. Williams and S. 2012)

There is an equivariant bijection between the order ideals of $\mathbf{A}_b$*, the triangular poset with* $b-1$ *minimal elements, under rowmotion and* $2 \times b$ *standard Young tableaux under promotion. So* $(J(\mathbf{A}_b), f, \mathrm{Row})$ *exhibits the cyclic sieving phenomenon, where* $f(q) = \frac{(2n)!_q}{(n+1)!_q n!_q}$.



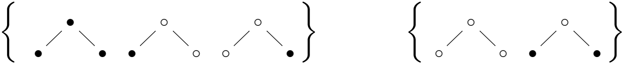
1	2	3
4	5	6

1	2	5
3	4	6

1	3	4
2	5	6

1	3	5
2	4	6

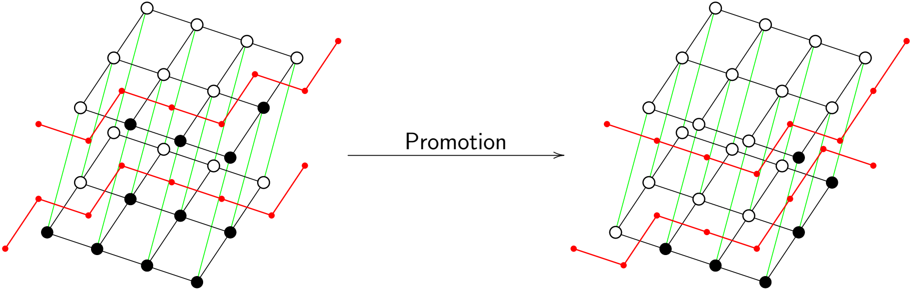
1	2	4
3	5	6



Cyclic sieving of rowmotion in $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$

Theorem (N. Williams and S. 2012)

There is an equivariant bijection between order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ *under rowmotion and noncrossing partitions of* $a + b + 1$ *into* $b + 1$ *blocks under rotation. So* $(J(\mathbf{a} \times \mathbf{b} \times \mathbf{2}), f, \mathrm{Row})$ *exhibits the cyclic sieving phenomenon, where* $f(q) = \displaystyle\prod_{1 \le i \le a, 1 \le j \le b, 1 \le k \le 2} \frac{[i+j+k-1]_q}{[i+j+k-2]_q}$.



Cyclic sieving of rowmotion in $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$

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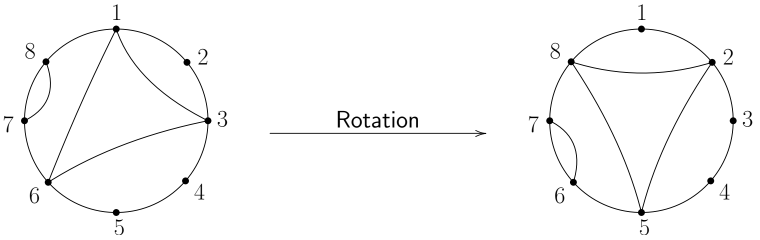
There is an equivariant bijection between order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ *under rowmotion and noncrossing partitions of* $a+b+1$ *into* $b+1$ *blocks under rotation. So* $(J(\mathbf{a} \times \mathbf{b} \times \mathbf{2}), f, \mathrm{Row})$ *exhibits the cyclic sieving phenomenon, where* $f(q) = \prod_{1 \le i \le a, 1 \le j \le b, 1 \le k \le 2} \frac{[i+j+k-1]_q}{[i+j+k-2]_q}$.



Cyclic sieving of rowmotion

Cyclic sieving also occurs with respect to rowmotion in the following posets:

- Minuscule posets ($\mathbf{a} \times \mathbf{b}$ is Type A) and Minuscule $\times$ $\mathbf{2}$ (D. Rush, X. Shi 2013)
  $\rightarrow$ Cyclic sieving does not occur in general for $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$. But it does occur for some other Minuscule $\times$ $\mathbf{c}$ (H. Mandel, O. Pechenik 2017+)
- Positive root posets ($\mathbf{A}_b$ is type A) (D. Armstrong, C. Stump, H. Thomas 2013)

Cyclic sieving has also been found in actions on tableaux, binary strings, permutations, noncrossing partitions, triangulations, multisets, matrices (Sagan survey 2011).

Cyclic sieving in Sage

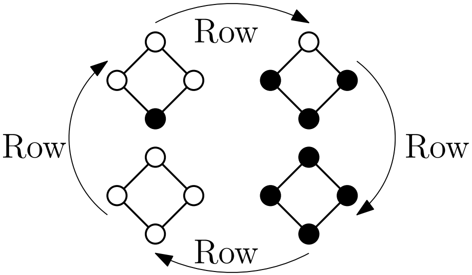
Rowmotion and toggle dynamics in Sage

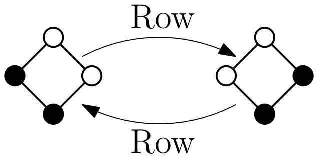
1. Promotion and rowmotion
   - Posets and order ideals
   - Rowmotion
   - Toggles
2. Cyclic sieving
3. Homomesy
4. Multidimensional promotion and rowmotion
5. Increasing labeling promotion and rowmotion

The homomesy phenomenon

Definition (J. Propp and T. Roby 2015)

A statistic on a set exhibits *homomesy* with respect to an action when the orbit-average of the statistic equals the global average of that statistic.





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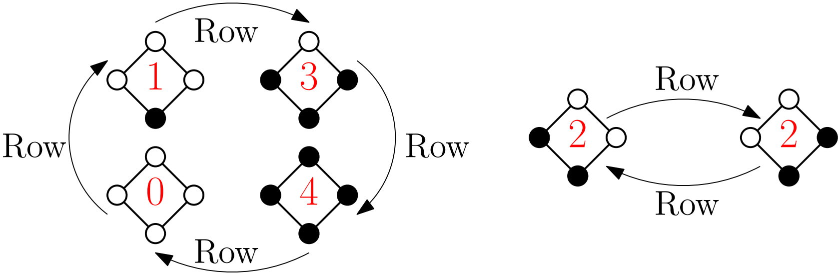
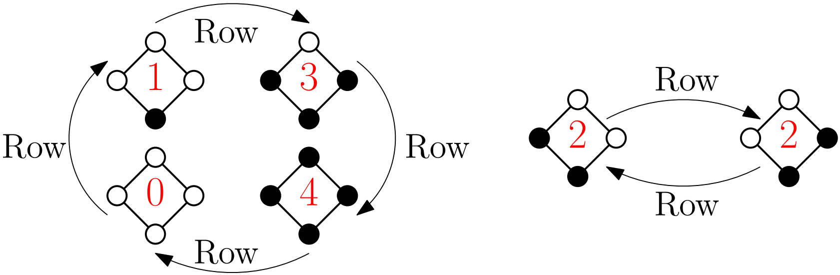


$$\frac{0+1+3+4}{4} = 2 \qquad \frac{2+2}{2} = 2$$

Homomesy of rowmotion in $\mathbf{a} \times \mathbf{b}$

Theorem (J. Propp and T. Roby 2015)

The cardinality statistic on order ideals of $\mathbf{a} \times \mathbf{b}$ exhibits homomesy (orbit-average = global-average) with respect to rowmotion.

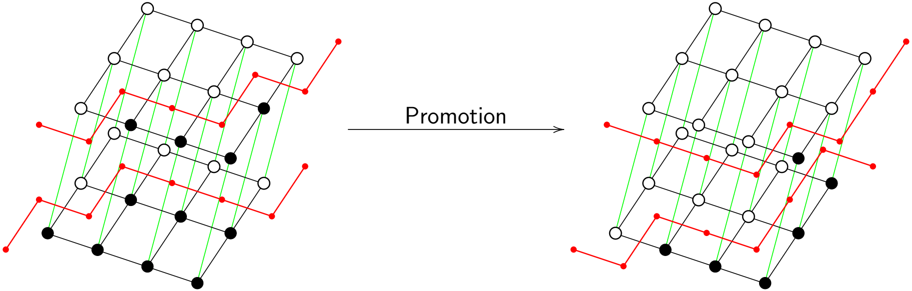


$$\frac{0+1+3+4}{4} = 2 \qquad \frac{2+2}{2} = 2$$

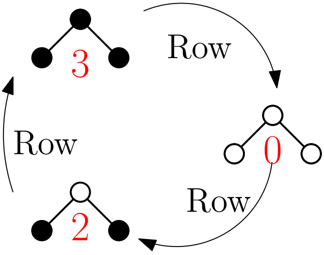
Homomesy of rowmotion in $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$

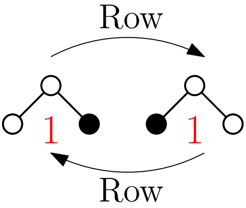
Theorem (C. Vorland 2018+)

The cardinality statistic on order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ *exhibits homomesy (orbit-average = global-average) with respect to rowmotion.*



Homomesy of rowmotion in triangular posets $\mathbf{A}_b$



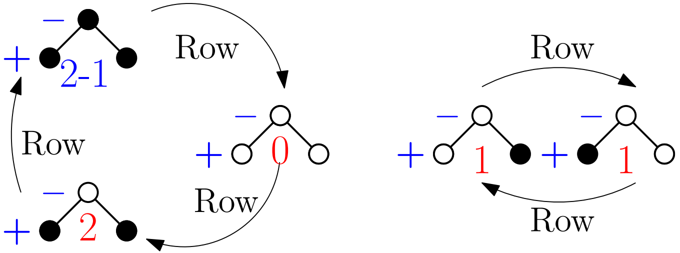


$$\frac{0+2+3}{3} = \frac{5}{3} \qquad \frac{1+1}{2} = 1$$

Homomesy of rowmotion in triangular posets $\mathbf{A}_b$

Theorem (S. Hadaddan)

The signed cardinality statistic on order ideals of $\mathbf{A}_b$ exhibits homomesy (orbit-average = global-average) with respect to rowmotion.



$$\frac{0+2+1}{3} = 1 \qquad \frac{1+1}{2} = 1$$

Homomesy of rowmotion

Homomesy occurs with respect to rowmotion in:

- Minuscule posets ($\mathbf{a} \times \mathbf{b}$ is Type A) (Rush and Wang)
- $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ and Type B Minuscule $\times$ $\mathbf{2}$ (Vorland)
  $\rightarrow$ Homomesy does not occur in general for $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.
- Triangular posets $\mathbf{A}_b$ with a *signed* cardinality statistic (Haddadan)
- Positive root posets ($\mathbf{A}_b$ is type A) with the *antichain* cardinality statistic (Armstrong, Stump, Thomas)

Homomesy has also been found in actions on independent sets, tableaux, binary strings, permutations, noncrossing partitions, vector spaces and simple harmonic motion (Roby survey 2016).

Homomesy in Sage

Rowmotion and toggle dynamics in Sage

1. Promotion and rowmotion
   - Posets and order ideals
   - Rowmotion
   - Toggles
2. Cyclic sieving
3. Homomesy
4. Multidimensional promotion and rowmotion
5. Increasing labeling promotion and rowmotion

Promotion and rowmotion are conjugate actions

Theorem (N. Williams and S. 2012)

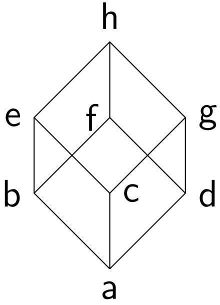
In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

Question: What do we mean by 'left-to-right'?

Multidimensional promotion and rowmotion

Definition

A **lattice projection** of a poset P is an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, where $x \leq y$ in $\mathbb{Z}^n$ if and only if the component-wise difference $y - x$ is in $\mathbb{N}^n$.



Multidimensional promotion and rowmotion

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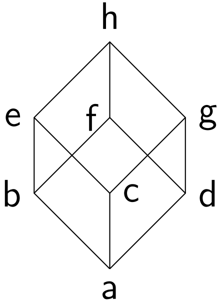

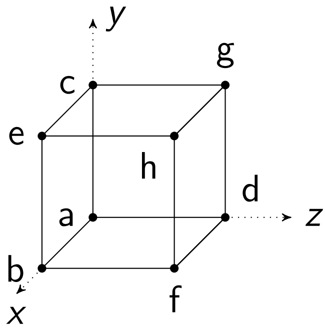

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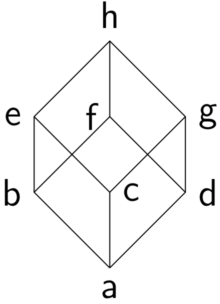
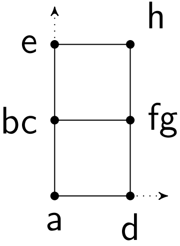
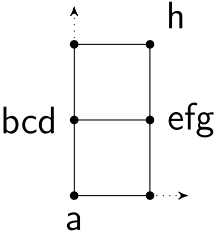






Multidimensional promotion and rowmotion

Definition

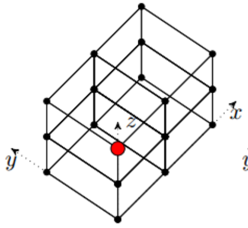
Let P be a poset with an n-dimensional lattice projection π, and let v be a vector with entries in $\{\pm 1\}$. Let $T_{\pi,v}^{i}$ be the product of toggles t_x for all elements x of P that lie on the affine hyperplane $\langle \pi(x), v\rangle = i$. Then define **promotion with respect to** π **and** v as

$$\underset{\pi,v}{\mathsf{Pro}} = \dots T_{\pi,v}^{-2} T_{\pi,v}^{-1} T_{\pi,v}^{0} T_{\pi,v}^{1} T_{\pi,v}^{2} \dots .$$

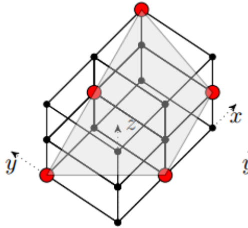
Proposition

For any finite ranked poset P and lattice projection π, $\mathsf{Pro}_{\pi,(1,1,\dots,1)} = \mathrm{Row}$.

$$\mathrm{Pro}_{\mathrm{id},(1,1,-1)} = T_{\pi,v}^{-2} T_{\pi,v}^{-1} T_{\pi,v}^{0} T_{\pi,v}^{1} T_{\pi,v}^{2} T_{\pi,v}^{3}$$



$x + y - z = -2$ $x + y - z = -1$ $x + y - z = 0$



$x + y - z = 1$ $x + y - z = 2$ $x + y - z = 3$

Multidimensional promotion and rowmotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

Let P be a finite poset with an n-dimensional lattice projection π. Let v and w be vectors with entries in $\{\pm 1\}$. Then there is an equivariant bijection between the order ideals under $\mathrm{Pro}_{\pi,v}$ and $\mathrm{Pro}_{\pi,w}$.

Rowmotion and $2^n - 1$ other promotions have the same orbit structure!

What is the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$?

Recall from earlier in the talk:

- The order of rowmotion on $\mathbf{a} \times \mathbf{b}$ is $a + b$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ is $a + b + 1$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{3}$ is $a + b + 2$ for as high as we can check.
- This may lead one to conjecture that the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ is $a + b + c - 1$.
- But the orbits of rowmotion on $\mathbf{4} \times \mathbf{4} \times \mathbf{4}$ are of size 11 and 33.

Increasing tableaux

Definition

An **increasing tableau** is a filling of a partition shape with positive integers that strictly increase from left to right across rows and from top to bottom down columns.

1	4	5	8
2	5	7	9
6	7	9	10
8	10		

K-Promotion on as a product of involutions

Proposition (K. Dilks, O. Pechenik, S. 2017)

K-Promotion is the product $\prod_i K\text{-BK}_i$, where $K\text{-BK}_i$ increments i and/or decrements $i+1$ **when possible**.

1	2	4	6
4	5	6	7

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K-Promotion on as a product of involutions

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1	3	5	6
3	4	6	7

Resonance of K-Promotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

Increasing tableaux under K-promotion exhibits **resonance***, that is, there is a projection from an increasing tableau to its binary content vector such that K-promotion maps to a cyclic shift.*

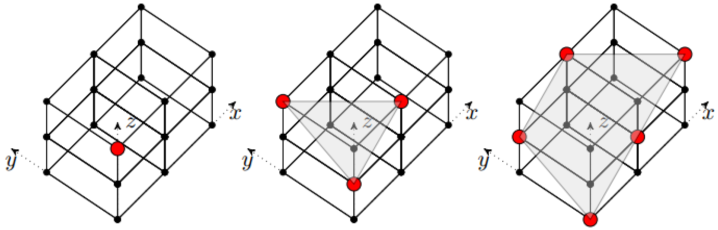
$$\begin{tikzcd}
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 8 \\ \hline 5 & 7 & 8 & 10 \\ \hline 7 & 9 & 10 & 12 \\ \hline \end{array} \arrow[r, "K\text{-Promotion}"] \arrow[d, "\mathrm{Con}"] & \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 7 & 9 \\ \hline 4 & 6 & 9 & 11 \\ \hline 6 & 8 & 11 & 12 \\ \hline \end{array} \arrow[d, "\mathrm{Con}"] \\
(1,1,1,1,1,1,1,1,1,1,0,1) \arrow[r, "\text{Cyclic shift}"] & (1,1,1,1,1,1,1,1,1,0,1,1)
\end{tikzcd}$$

An equivariant bijection between increasing tableaux and order ideals in $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$

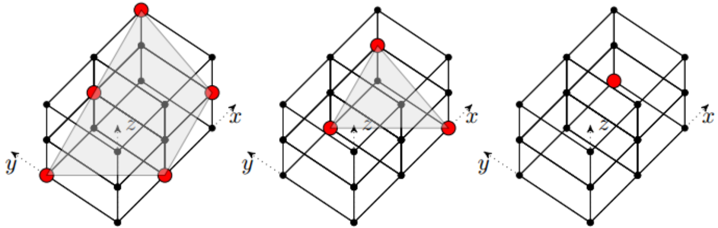
Theorem (K. Dilks, O. Pechenik, S. 2017)

Order ideals in $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ under rowmotion are in equivariant bijection with increasing tableaux of rectangular shape $a \times b$ and entries at most $a + b + c - 1$ under K-promotion.

K-Bender-Knuth involutions correspond to toggling by hyperplanes



$x + y - z = -2$ $\quad$ $x + y - z = -1$ $\quad$ $x + y - z = 0$



Resonance of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$

Theorem (K. Dilks, O. Pechenik, S. 2017)

Rowmotion on order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ *exhibits* **resonance with frequency** $a + b + c - 1$, *that is, there is a projection from an order ideal in* $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ *through the corresponding increasing tableau to its binary content vector such that rowmotion maps to a cyclic shift.*

So even though rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ does not have order $a + b + c - 1$, it projects to something that does!

Rowmotion and toggle dynamics in Sage

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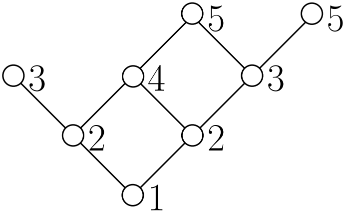
Promotion on increasing labelings of a poset

An increasing tableau is an **increasing labeling** of a partition shaped poset.

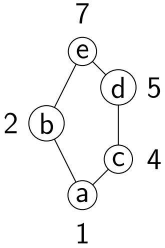
1	2	3	5
2	4	5	
3			

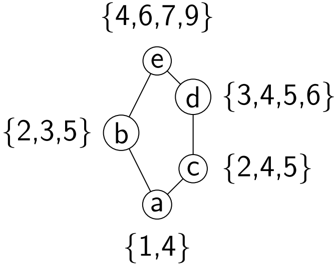


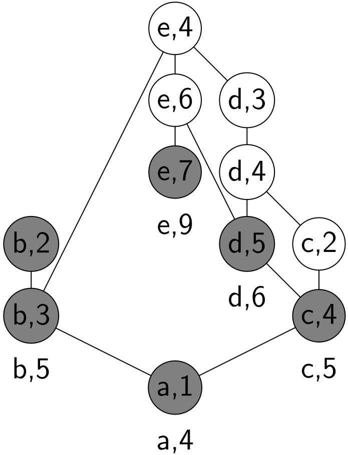
We may define generalized promotion on increasing labelings of any poset by *generalized Bender-Knuth involutions*.

Theorem (K. Dilks, S., C. Vorland 2018+)

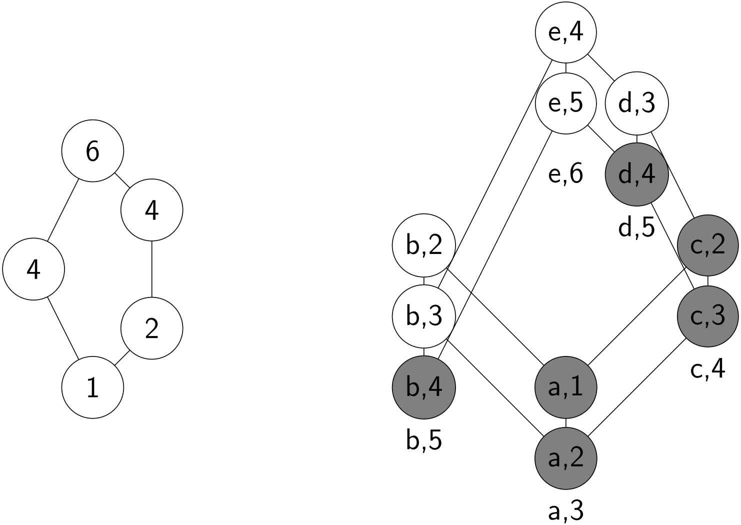
Increasing labelings $\mathrm{Inc}^R(P)$ *under* IncPro *are in equivariant bijection with order ideals of* $\Gamma(P,R)$ *under* $\mathrm{TogPro}_{H_\Gamma}$.



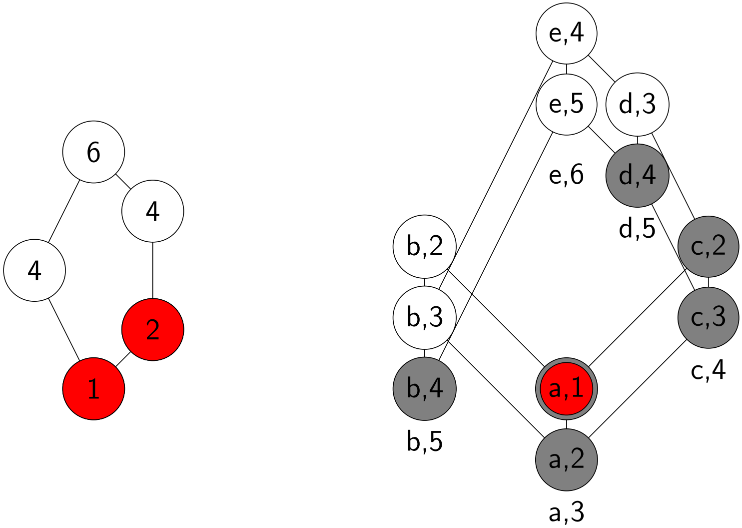




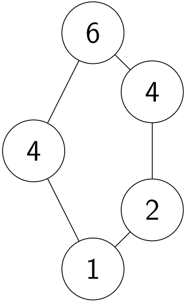
Each ρ_i involution corresponds to a generalized toggle T_H^i

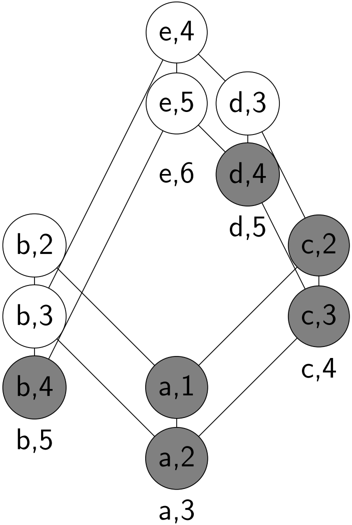


Each ρ_i involution corresponds to a generalized toggle T_H^i



Each ρ_i involution corresponds to a generalized toggle T_H^i





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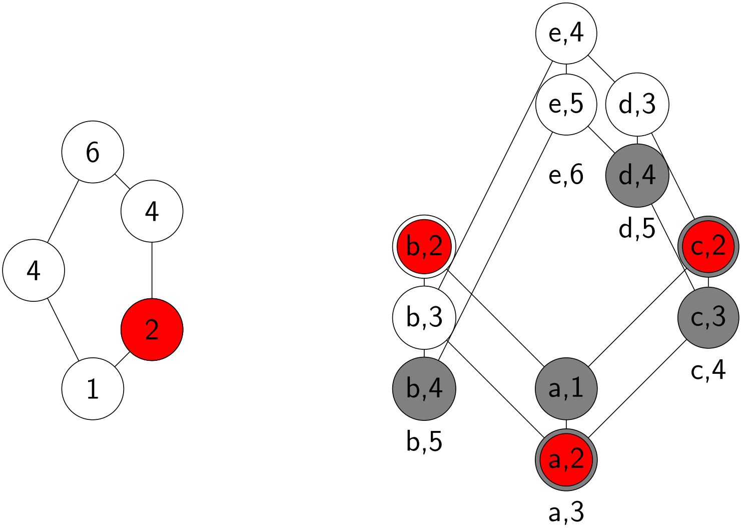

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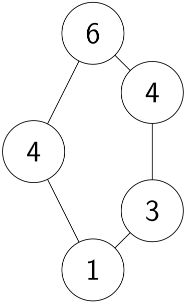

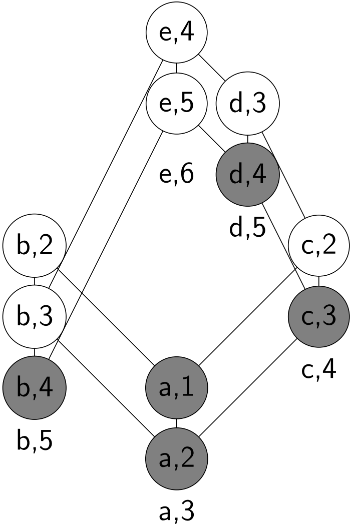

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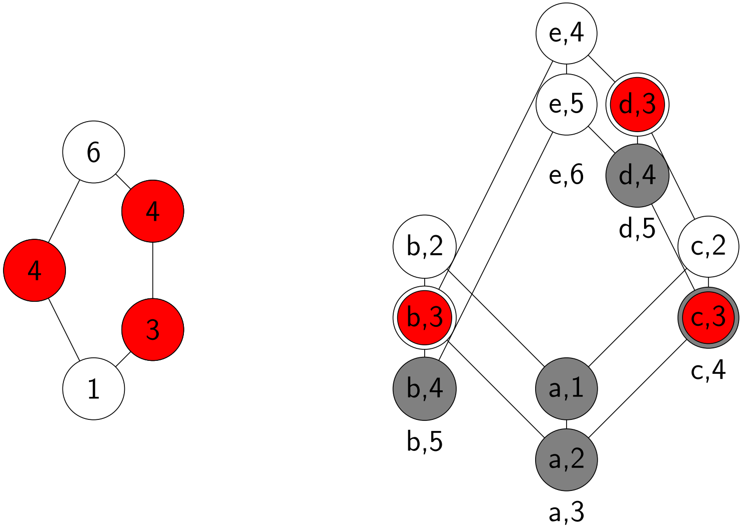

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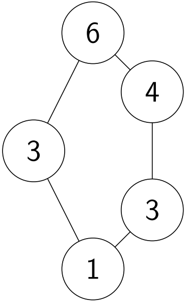

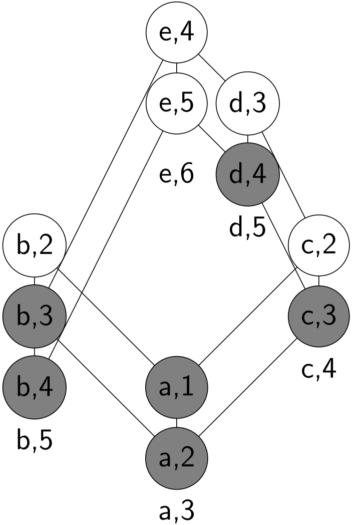

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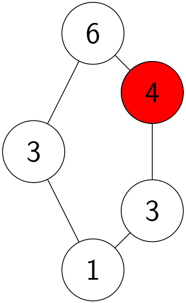

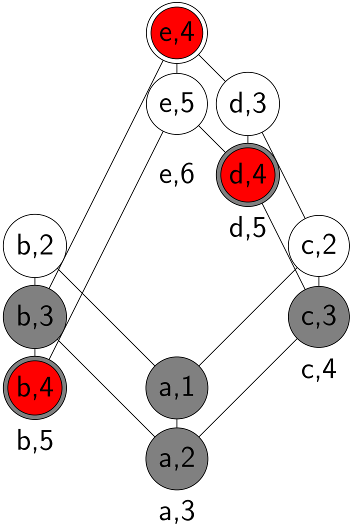

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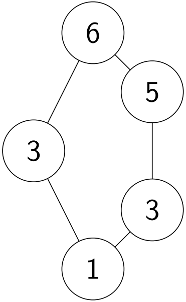

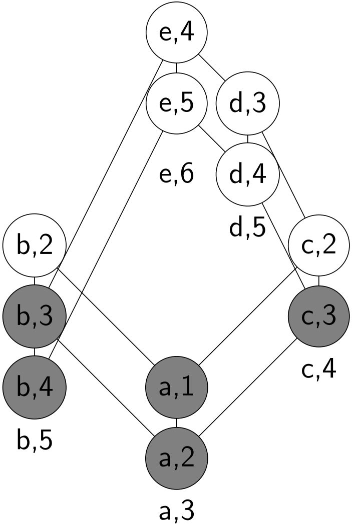

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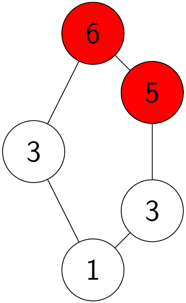

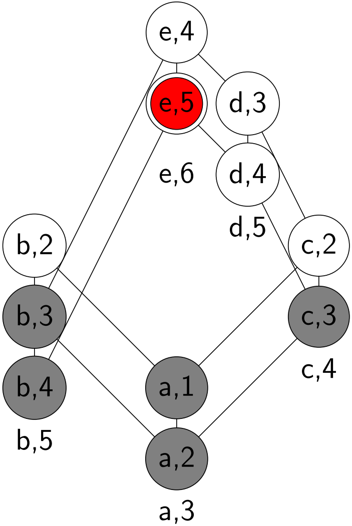

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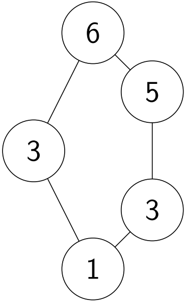

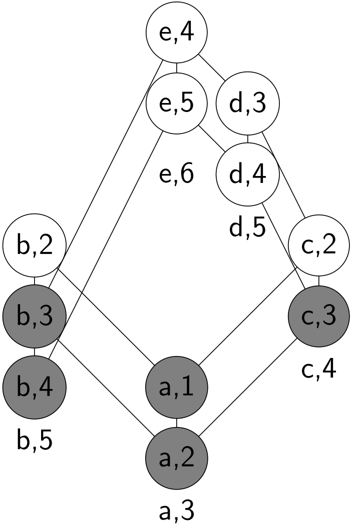

How does TogPro relate to rowmotion?

Definition

$H : P \to \mathbb{Z}$ is a *column toggle order* if whenever $p_1 \lessdot p_2$ in P, then $H(p_1) = H(p_2) \pm 1$.

Theorem (K. Dilks, S., C. Vorland 2018+)

When H_Γ is a column toggle order, there is an equivariant bijection between $\mathrm{Inc}^R(P)$ under IncPro *and order ideals of $\Gamma(P, R)$ under Row.*

Corollary

There is an equivariant bijection between $\mathrm{Inc}^q(P)$ under IncPro *and order ideals of $\Gamma(P, q)$ under Row.*

Rowmotion and toggle dynamics in Sage

1. Promotion and rowmotion
   - Posets and order ideals
   - Rowmotion
   - Toggles
2. Cyclic sieving
3. Homomesy
4. Multidimensional promotion and rowmotion
5. Increasing labeling promotion and rowmotion

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