Use of Sage in my Research–Two Examples

Kyu-Hwan Lee

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1. First Example

– Joint work with Jang Soo Kim and Se-jin Oh

- $\mathfrak{g}$: affine Kac–Moody algebra of classical type $X_n^{(r)}$,

$$(X = A, B, C, D \text{ and } r = 1, 2)$$

  $\Lambda \in P^+$: dominant integral weight

  $V(\Lambda) = \bigoplus_\mu V(\Lambda)_\mu$: irreducible highest weight module

- A weight λ of $V(\Lambda)$ is called *maximal* if $\lambda + \delta$ is not a weight of $V(\Lambda)$, where δ is the null root. We have

$$V(\Lambda) = \bigoplus_{\lambda \in \max(\Lambda)} \bigoplus_{k=0}^{\infty} V(\Lambda)_{\lambda - k\delta},$$

  where $\max(\Lambda)$ is the set of maximal weights of $V(\Lambda)$.

- It is enough to consider $\max(\Lambda) \cap P^+$.
- **Problem**: Determine multiplicities $\dim V(\Lambda)_\lambda$ for $\lambda \in \max(\Lambda) \cap P^+$.
- There are algorithms to compute each individual multiplicity. But we want to have a systematic description.
- Moreover, these multiplicities are also weight multiplicities of the finite dimensional Lie algebras of type X_n.

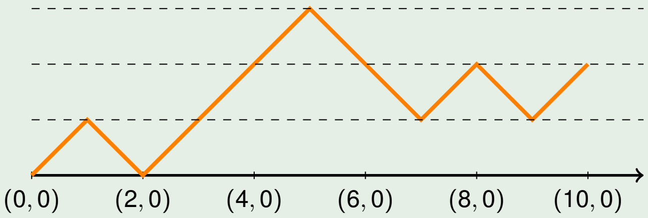
1.1. Catalan triangle

Definition

A *Dyck path* is a path on the lattice $\mathbb{Z}^2$ starting from $(0,0)$, using steps $(1,1)$, $(1,-1)$ without going below the x-axis.

Example

The following path is a Dyck path from $(0,0)$ to $(10,2)$:



A *Catalan triangle number* $\mathrm{C}_{(m,k)}$ for $m \geq k \geq 0$ is the number of all Dyck paths ending at the lattice point (m, k). We form a triangular array consisting of $\mathrm{C}_{(m,k)}$, called the *Catalan triangle*.

											⋰
										1	⋯
									1	0	⋯
								1	0	9	⋯
							1	0	8	0	⋯
						1	0	7	0	35	⋯
					1	0	6	0	27	0	⋯
				1	0	5	0	20	0	75	⋯
			1	0	4	0	14	0	48	0	⋯
		1	0	3	0	9	0	28	0	90	⋯
	1	0	2	0	5	0	14	0	42	0	⋯
1	0	1	0	2	0	5	0	14	0	42	⋯

Theorem (Tsuchioka, 2009)

For type $A_n^{(1)}$, $n \geq m$, each nonzero $\mathsf{C}_{(m,k)}$ is the multiplicity of a maximal weight for the highest weight $\Lambda_0 + \Lambda_k$, $k = 0, 1, \ldots, m$.

- Can we have similar results for other types?
- We need to see data of weight multiplicities.
- The SageMath package developed by Daniel Bump was extremely helpful.
- After extensive experiments, we could see patterns.

Theorem (Kim-L.-Oh)

For types $C_n^{(1)}$, $n \geq m$, each nonzero $\mathsf{C}_{(m,k)}$ is the multiplicity of a maximal weight for the highest weight Λ_k, $k = 0, 1, \ldots, m$.

- In particular, highest weights $\Lambda_0 + \Lambda_k$ of $A_n^{(1)}$ and Λ_k of $C_n^{(1)}$ have the same maximal weight multiplicities.

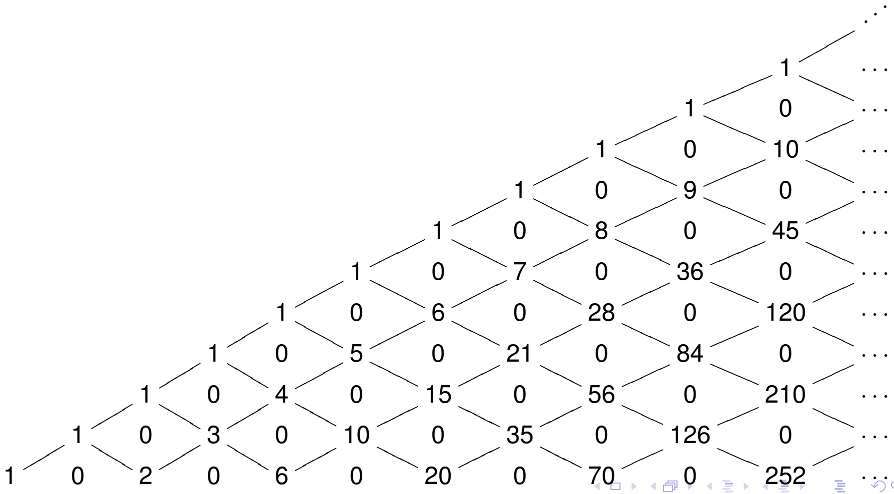
Definition

A *binomial path* is a path on the lattice $\mathbb{Z}^2$ starting from $(0, 0)$, using steps $(1, 1)$, $(1, -1)$.

Example

The following path is a binomial path from $(0, 0)$ to $(10, 2)$:

A *binomial coefficient* $\mathsf{B}_{(m,k)}$ for $m \geq k \geq 0$ is the number of all binomial paths ending at the lattice point (m, k). Then we can form the *Pascal triangle* consisting of $\mathsf{B}_{(m,k)}$.



Theorem (Kim-L.-Oh)

For types $B_n^{(1)}$, $D_n^{(1)}$ and $A_{2n-1}^{(2)}$, $n \geq m$, each nonzero $\mathsf{B}_{(m,k)}$ is the multiplicity of a maximal weight for the highest weight $\Lambda_k + \delta_{k,1}\Lambda_0$, $k = 1, 2, \ldots, m$.

- We recognize combinatorics of crystals behind the triangle.

  Lattice path $\Longleftrightarrow$ tableaux

  $\Longleftrightarrow$ crystal element in the crystal graph

- The triangle can be understood through an insertion scheme.

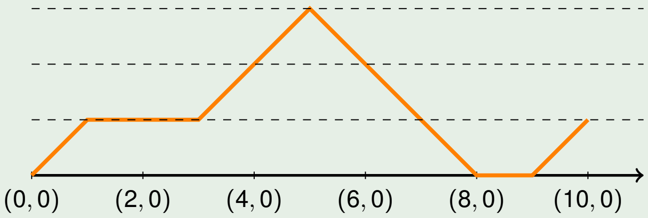
1.3. Motzkin triangle

Definition

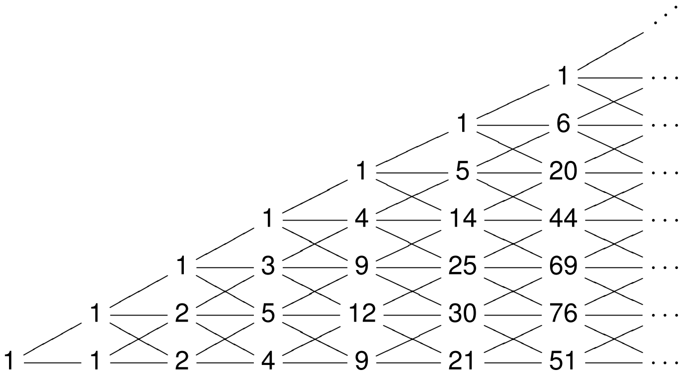
A *Motzkin path* is a path on the lattice $\mathbb{Z}^2$ starting from $(0,0)$, using steps $(1,1)$, $(1,0)$, $(1,-1)$ without going below the x-axis.

Example

The following path is a Motzkin path from $(0,0)$ to $(10,1)$:



A *generalized Motzkin number* $M_{(m,k)}$ for $m \geq k \geq 0$ is the number of all Motzkin paths ending at the lattice point (m, k). Then we can form a triangular array consisting of $M_{(m,k)}$, called the *Motzkin triangle*.



Theorem (Kim-L.-Oh)

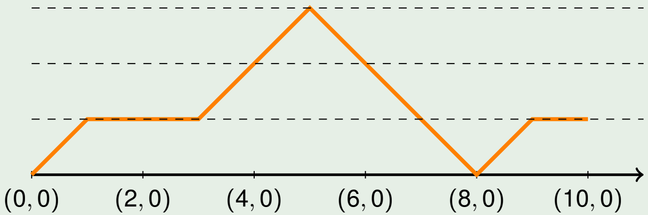
For types $B_n^{(1)}$ (resp. $A_{2n}^{(2)}$ and $D_{n+1}^{(2)}$), $n \geq m$, each nonzero $\mathrm{M}_{(m,k)}$ is the multiplicity of a maximal weight for the highest weight $\Lambda_n + \Lambda_{n-k} + \delta_{k,0}\Lambda_n$ (resp. $\Lambda_0 + \Lambda_k + \delta_{k,0}\Lambda_0$), $k = 0, 1, \ldots, m$.

Definition

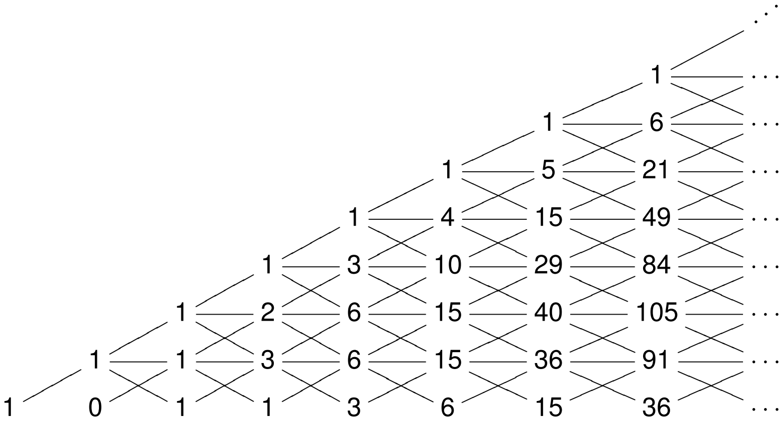
A *Riordan path* is a Motzkin path which has no horizontal step on the x-axis.

Example

The following path is a Riordan path:



A *generalized Riordan number* $\mathrm{R}_{(m,s)}$ for $m \geq s \geq 0$ is the number of all Riordan paths ending at the lattice point (m, s). Then we can form a triangular array consisting of $\mathrm{R}_{(m,s)}$, called the *Riordan triangle*.



Theorem (Kim-L.-Oh)

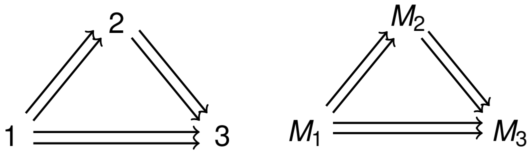
For types $B_n^{(1)}$, $D_n^{(1)}$ and $A_{2n-1}^{(2)}$, $n \geq m$, each nonzero $\mathrm{R}_{(m,k)}$ is the multiplicity of a maximal weight for the highest weight $\Lambda_0 + \Lambda_k + \delta_{k,1}\Lambda_0$, $k = 1, 2, \ldots, m$.

2. Second Example

– Joint work with Kyungyong Lee

- $\mathcal{Q}$: acyclic quiver of rank N
- $M \in \mathrm{rep}(\mathcal{Q})$: representation of $\mathcal{Q}$ over $\mathbb{C}$

  When $N = 3$,



- $d_M = (\dim M_1, \ldots, \dim M_N)$: dimension vector of M

- For $M, N \in \mathrm{rep}(\mathcal{Q})$, define

$$\langle d_M, d_N \rangle = \dim \mathrm{Hom}(M, N) - \dim \mathrm{Ext}^1(M, N)$$

  and

$$(d_M, d_N) = \langle d_M, d_N \rangle + \langle d_N, d_M \rangle.$$

- When M is indecomposable, we call d_M a (positive) root.
- A root α is called real if $(\alpha, \alpha) = 2$.
- A real root α is called a real Schur root if $\alpha = d_M$ and $\mathrm{Ext}^1(M, M) = 0$.

– We can use the Weyl group to generate real roots.

- Define the simple roots

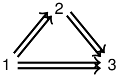
  $\alpha_1 = (1, 0, \ldots, 0)$, $\alpha_2 = (0, 1, 0, \ldots, 0)$, ..., $\alpha_N = (0, \ldots, 0, 1)$.

- Define $s_i : \mathbb{Z}^N \longrightarrow \mathbb{Z}^N$ by

$$s_i(\beta) = \beta - (\beta, \alpha_i)\alpha_i.$$

  The Weyl group W is the subgroup of $GL_N(\mathbb{Z})$ generated by s_i, $i = 1, 2, \ldots, N$.

- A root α is real $\Longleftrightarrow \exists w \in W$ such that $\alpha = w\alpha_i$ for some i.

Let $N = 3$, and $\mathcal{Q}$ be the quiver 

.

$$\alpha_1 + 6\alpha_2 + 2\alpha_3 = s_2 s_3 \alpha_1,$$

$$15\alpha_1 + 6\alpha_2 + 2\alpha_3 = s_1 s_2 s_3 \alpha_1,$$

$$2385\alpha_1 + 924\alpha_2 + 340\alpha_3 = (s_1 s_2 s_3)^2 s_2 s_3 s_2 s_3 \alpha_1,$$

$$1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3 = (s_3 s_2 s_1)^4 s_2 s_3 s_2 s_1 s_2 s_3 \alpha_2.$$

Question: Which ones are real Schur roots?

- There are answers due to Schofield, Speyer, Thomas, ...

Problem: Give a description for real Schur roots which can be used to distinguish real Schur roots among all real roots.

– We can use cluster variables to generate real Schur roots.

Theorem (Fomin–Zelevinsky, 2002)

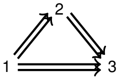
Each cluster variable is a Laurent polynomial over $\mathbb{Z}$ in the initial cluster variables $x_1, x_2, \ldots, x_N$.

Therefore, the denominator of every cluster variable is well defined and we have

$$x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N} \longmapsto (m_1, m_2, \ldots, m_N)$$
$$= m_1\alpha_1 + m_2\alpha_2 + \cdots + m_N\alpha_N.$$

Theorem (Caldero–Keller, 2006)

The above correspondence is a bijection between the set of denominators of non-initial cluster variables and the set of real Schur roots of $\mathcal{Q}$.

For example, 

$$\Theta = \left((x_1, x_2, x_3), \begin{pmatrix} 0 & 2 & 2 \\ -2 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix} \right)$$

$$\mu_2\mu_1\mu_3\mu_2(\mu_1\mu_2\mu_3)^4\Theta$$

$$= \left(\left(\frac{P_1}{x_1^{167041} x_2^{437340} x_3^{1144950}}, \frac{P_2}{x_1^{1662490} x_2^{4352663} x_3^{11395212}}, \frac{P_3}{x_1^{28656} x_2^{75026} x_3^{196417}} \right), \begin{pmatrix} 0 & 10 & -574 \\ -10 & 0 & 58 \\ 574 & -58 & 0 \end{pmatrix} \right)$$

We obtain three real Schur roots:

$$167041\alpha_1 + 437340\alpha_2 + 1144950\alpha_3,$$

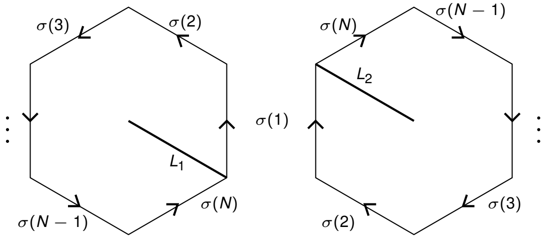
$$1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3,$$

$$28656\alpha_1 + 75026\alpha_2 + 196417\alpha_3.$$

- There is a SageMath package for computing cluster variables.
- Can we use this package to generate real Schur roots?
- The package calculates numerators and denominators of cluster variables.
- Instead, we needed to write a code for denominators only.
- After some experiments, we could formulate a conjecture.

2.1. Conjecture

- S_N: the permutation group on $I := \{1, 2, \ldots, N\}$
- Let $P_{\mathcal{Q}} \subset S_N$ be the set of all permutations σ such that there is no arrow from $\sigma(j)$ to $\sigma(i)$ for any $j > i$ on $\mathcal{Q}$.
- For each $\sigma \in P_{\mathcal{Q}}$, consider



- Let Σ_σ be the compact Riemann surface of genus $\lfloor \frac{N-1}{2} \rfloor$ obtained by gluing together the two N-gons with all the edges of the same label identified according to their orientations.
- The edges of the N-gons become N different curves in Σ_σ.
- If N is odd, there is one vertex on Σ_σ.

  If N is even, there are two vertices on Σ_σ.
- $\mathcal{T} := T_1 \cup \cdots T_N \subset \Sigma_\sigma$

  V: the set of the vertex (or vertices) on $\mathcal{T}$
- Define R to be the set of words $i_1 i_2 \cdots i_k$ ($i_p \neq i_{p+1}$) such that $s_{i_1} s_{i_2} \cdots s_{i_k}$ is a reflection in W.

For $\sigma \in P_{\mathcal{Q}}$, define a σ-admissible curve $\eta : [0,1] \longrightarrow \Sigma_\sigma$ by

1. $\eta(x) \in V$ if and only if $x \in \{0,1\}$;
2. there exists $\epsilon > 0$ such that $\eta([0,\epsilon]) \subset L_1$ and $\eta([1-\epsilon,1]) \subset L_2$;
3. if $\eta(x) \in \mathcal{T} \setminus V$ then $\eta([x-\epsilon, x+\epsilon])$ meets $\mathcal{T}$ transversally for sufficiently small $\epsilon > 0$;
4. $\upsilon(\eta) \in R$, where $\upsilon(\eta) := i_1 \cdots i_k$ is given by

$$\{x \in [0,1] \ : \ \eta(x) \in \mathcal{T}\} = \{x_1 < \cdots < x_k\} \quad \text{and}$$

$$\eta(x_\ell) \in T_{i_\ell} \text{ for } \ell \in \{1, ..., k\}.$$

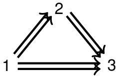
- We have

$$\eta \rightsquigarrow \upsilon(\eta) \rightsquigarrow w \rightsquigarrow \beta(\eta)$$

Let Γ_σ be the set of (isotopy classes of) σ-admissible curves η with no self-intersections.

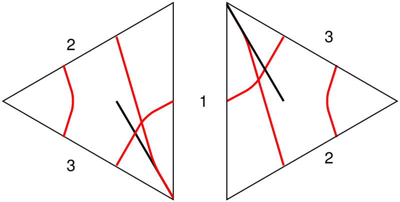
Conjecture

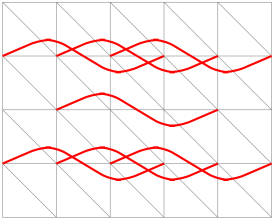
For each $\eta \in \Gamma_\sigma$, let $\beta(\eta)$ be the positive real root corresponding to η. Then $\{\beta(\eta) \ : \ \eta \in \cup_{\sigma \in P_{\mathcal{Q}}} \Gamma_\sigma\}$ is precisely the set of real Schur roots for $\mathcal{Q}$.

For example, 

Then $P_{\mathcal{Q}} = \{id\}$.

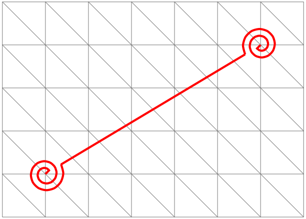
(1) We have $v(\eta) = 23132$ for the admissible curve η given by





The corresponding reflection is $w = s_2 s_3 s_1 s_3 s_2$, and its real root is $s_2 s_3 \alpha_1 = \alpha_1 + 6\alpha_2 + 2\alpha_3$. Since the curve η has self-intersections, this real root is not Schur (if the conjecture is true).

(2) We get $\upsilon(\eta) = (321)^4 2321232321232(123)^4$ for η given by



The corresponding reflection is

$w = (s_3 s_2 s_1)^4 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_3 s_2 s_1 s_2 s_3 s_2 (s_1 s_2 s_3)^4$, and its real root is

$$(s_3 s_2 s_1)^4 s_2 s_3 s_2 s_1 s_2 s_3 \alpha_2 = 1662490\alpha_1 + 4352663\alpha_2 + 11395212\alpha_3.$$

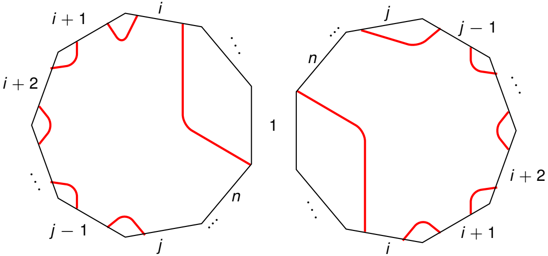
Since the curve η has no self-intersections, this real root is Schur (if the conjecture is true).

- We need to check the conjecture. We wrote a code to compute $\beta(\eta)$ from η for rank 3 quivers.
- We prove the conjecture for 2-complete quivers of rank 3.
- Felikson and Tumarkin proved the conjecture for all 2-complete quivers of arbitrary ranks.
- The conjecture is wide open for general quivers.
- For Dynkin quivers, the conjecture can be checked.

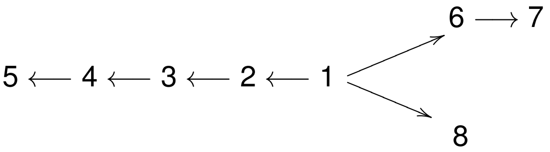
— Consider type A Dynkin quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

- All positive real roots are Schur.
- Each positive real root is equal to $s_i s_{i+1} \cdots s_{j-1}\alpha_j$, and the corresponding reflection is $w = s_i \cdots s_{j-1} s_j s_{j-1} \cdots s_i$.
- There exists an admissible curve η on Σ_{id} with no self-intersections and $v(\eta) = i \cdots (j-1)j(j-1) \cdots i \in R_w$.



— Consider E_8 Dynkin quiver



- The highest positive real root

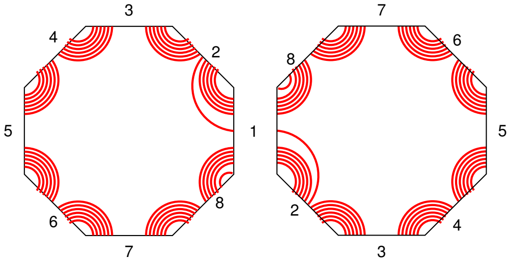
$$6\alpha_1 + 5\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

can be given by $(s_8 s_7 \cdots s_2 s_1)^5 (s_8 s_7 \cdots s_2)\alpha_1$.

- The corresponding reflection is

$$(s_8 s_7 \cdots s_2 s_1)^5 (s_8 s_7 \cdots s_2) s_1 (s_2 \cdots s_7 s_8)(s_1 s_2 \cdots s_7 s_8)^5.$$

- The root is given by the following non-self-intersecting curve on Σ_{id}.



Thank You