A Dynamic Near-Optimal Algorithm for Online Linear Programming

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Information-Based Complexity and Stochastic Computation

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Outline

- Online Linear Programming
- Main Results and Key Ideas
- Related and More Recent Work

Background

Consider a store that sells a number of goods/products

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- There is a fixed inventory of goods
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- Decision: To sell or not to sell to each individual customer?
- Objective: Maximize the revenue.

An Example

	Bid 1($t = 1$)	Bid 2($t = 2$)	.....	Inventory($\mathbf{b}$)
Price(π_t)	\$100	\$30	...	
Decision	x_1	x_2	...	
Pants	1	0	...	100
Shoes	1	0	...	50
T-shirts	0	1	...	500
Jackets	0	0	...	200
Hats	1	1	...	1000

Online Linear Programming Model

The classical offline version of the above program can be formulated as a linear (integer) program as all information data would have arrived: compute x_t, $t = 1, ..., n$ and

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\text{maximize}_{\mathbf{x}} & \sum_{t=1}^{n} \pi_t x_t & \\
\text{subject to} & \sum_{t=1}^{n} a_{it} x_t \leq b_i, & \forall i = 1, ..., m \\
& x_t \in \{0,1\} \ (0 \leq \mathbf{x}_t \leq 1), & \forall t = 1, ..., n.
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Now we consider the online or streamline and data-driven version of this problem:

- We only know $\mathbf{b}$ and n at the start
- the bidder information is revealed sequentially along with the corresponding objective coefficient.
- an irrevocable decision must be made as soon as an order arrives without observing or knowing the future data.

Application Overview

- Revenue management problems: Airline tickets booking, hotel booking;
- Online network routing on an edge-capacitated network;
- Online combinatorial auction;
- Online adwords allocation

Model Assumptions

Main Assumptions

- $0 \leq a_{it} \leq 1$, for all (i, t);
- $\pi_t \geq 0$ for all t
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Denote the offline LP maximal value by $OPT(A, \pi)$. We call an online algorithm $\mathcal{A}$ to be c-competitive if and only if

$$E_\sigma \left[\sum_{t=1}^{n} \pi_t x_t(\sigma, \mathcal{A}) \right] \geq c \cdot OPT(A, \pi) \ \forall (A, \pi),$$

where σ is the permutation of arriving orders.
In what follows, we let

$$B = \min_i \{b_i\} (> 0).$$

Main Results: Necessary and Sufficient Conditions

Theorem

For any fixed $0 < \epsilon < 1$, there is no online algorithm for solving the linear program with competitive ratio $1 - \epsilon$ if

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Agrawal, Wang and Y [Operations Research 2014]

Key Ideas: A Worst-Case Distribution Example

The proof of the negative result is based on a distribution of instances (the number of each types of columns is chosen according to certain distribution) with $m = 2^k$, and then show that no allocation rule can achieve $(1-\epsilon)$-optimality in expectation under randomized permutation.

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The proof of the positive result is constructive and based on a learning policy.

- There is no distribution known so that any type of stochastic optimization models is not applicable.
- Unlike dynamic programming, the decision maker does not have full information/data so that a backward recursion can not be carried out to find an optimal sequential decision policy.
- Thus, the online algorithm needs to be learning-based, in particular, learning-while-doing.

Price Observation of Online Learning I

The problem would be easy if there is an "ideal price" vector:

	Bid 1($t=1$)	Bid 2($t=2$)	.....	Inventory($\mathbf{b}$)	$\mathbf{p}^*$
Bid(π_t)	\$100	\$30	...		
Decision	x_1	x_2	...		
Pants	1	0	...	100	\$45
Shoes	1	0	...	50	\$45
T-shirts	0	1	...	500	\$10
Jackets	0	0	...	200	\$55
Hats	1	1	...	1000	\$15

Price Observation of Online Learning II

- Pricing the bid: The optimal dual price vector $\mathbf{p}^*$ of the offline LP problem can play such a role, that is $x_t^* = 1$ if $\pi_t > \mathbf{a}_t^T \mathbf{p}^*$ and $x_t^* = 0$ otherwise, yields a near-optimal solution.

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- One-time learning algorithm: learn the price vector once using the initial ϵn input.
- Dynamic learning algorithm: dynamically update the price vector at a carefully chosen pace.

One-Time Learning Algorithm

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  and get the optimal dual solution $\hat{\mathbf{p}}$;
- Determine the future allocation x_t as:

$$x_t = \begin{cases} 0 & \text{if } \pi_t \leq \hat{\mathbf{p}}^T \mathbf{a}_t \\ 1 & \text{if } \pi_t > \hat{\mathbf{p}}^T \mathbf{a}_t \end{cases}$$

  as long as $a_{it} x_t \leq b_i - \sum_{j=1}^{t-1} a_{ij} x_j$ for all i; otherwise, set $x_t = 0$.

One-Time Learning Algorithm Result

Theorem

For any fixed $\epsilon > 0$, the one-time learning algorithm is $(1-\epsilon)$ competitive for solving the linear program when

$$B \geq \Omega\left(\frac{m \log(n/\epsilon)}{\epsilon^3}\right)$$

Outline of the Proof

- With high probability, we clear the market;
- With high probability, the revenue is near-optimal if we include the initial ϵ portion revenue;
- With high probability, the first ϵ portion revenue, a learning cost, doesn't contribute too much.

Then, we prove that the one-time learning algorithm is $(1-\epsilon)$ competitive under condition $B \geq \frac{6m\log(n/\epsilon)}{\epsilon^3}$.

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But this is one ϵ factor higher than the lower bound...

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In the dynamic price learning algorithm, we update the price at time ϵn, $2\epsilon n$, $4\epsilon n$, ..., till $2^k\epsilon \geq 1$.

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At time $\ell \in \{\epsilon n, 2\epsilon n, \ldots\}$, the price vector is the optimal dual solution to the following linear program:

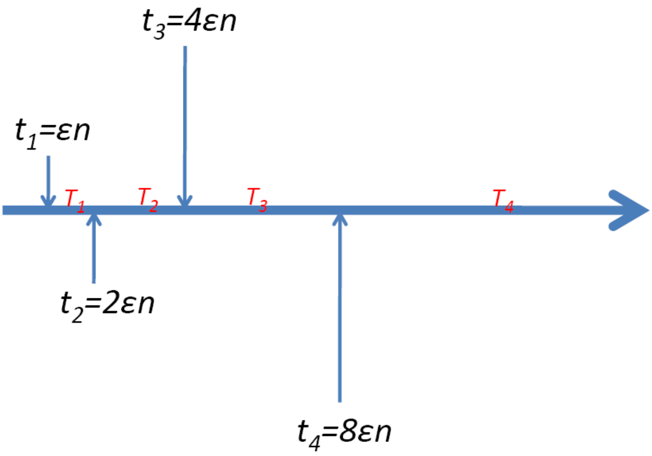
$$\begin{array}{lll} \text{maximize}_{\mathbf{x}} & \sum_{t=1}^{\ell} \pi_t x_t & \\ \text{subject to} & \sum_{t=1}^{\ell} a_{it} x_t \leq (1 - h_\ell)\frac{\ell}{n} b_i & i = 1, \ldots, m \\ & 0 \leq x_t \leq 1 & t = 1, \ldots, \ell \end{array}$$

where

$$h_\ell = \epsilon \sqrt{\frac{n}{\ell}};$$

and this price vector is used to determine the allocation for the next immediate period.

Geometric Pace/Grid of Price Updating



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- The numbers h_ℓ play an important role in improving the condition on B in the main theorem. It basically balances the probability that the inventory ever gets violated and the lost of revenue due to the factor $1 - h_\ell$.
- Choosing large h_ℓ (more conservative) at the beginning periods and smaller h_ℓ (more aggressive) at the later periods, one can now control the loss of revenue by an ϵ order while the required size of B can be weakened by an ϵ factor.

Related Work on Random-Permutation

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Kesselheim et al [2014]	$B \geq \frac{\log m}{\epsilon^2}$	Dynamic*
Gupta and Molinaro [2014]	$B \geq \frac{\log m}{\epsilon^2}$	Dynamic*

Table: Comparison of several existing results

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- price-posting market?