A reduced fast component-by-component construction of (polynomial) lattice points

Peter Kritzer
Johannes Kepler University Linz

Joint work with
J. Dick (Sydney), G. Leobacher (Linz), F. Pillichshammer (Linz)

Research supported by the Austrian Science Fund, Projects F5506-N26 and P23389-N18

Introduction and Motivation

Consider integration of functions on $[0,1]^s$,

$$I_s(f) = \int_{[0,1]^s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \, ,$$

where $f \in \mathcal{H}$, and $\mathcal{H}$ is some Banach space.

Approximate I_s by a QMC rule,

$$I_s(f) \approx Q_{N,s}(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(\boldsymbol{x}_k),$$

where $\mathcal{P}_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$.

Worst case error in Banach space $\mathcal{H}$ with respect to $\mathcal{P}_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$:

$$e_{N,s}(\mathcal{H}, \mathcal{P}_N) := \sup_{f \in \mathcal{H}, \|f\| \leq 1} \left| I_s(f) - Q_{N,s}(f) \right| .$$

Need $\mathcal{P}_N$ that makes $e_{N,s}(\mathcal{H}, \mathcal{P}_N)$ small.

Weighted Korobov space: $\mathcal{H}_{s,\alpha,\gamma}$ = space of continuous functions f such that $\|f\|_{s,\alpha,\gamma} < \infty$, where

$$\|f\|_{s,\alpha,\gamma}^2 = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} \rho_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})^{-1} \, |\widehat{f}(\boldsymbol{h})|^2,$$

and where $\widehat{f}(\boldsymbol{h}) = \int_{[0,1]^s} f(\boldsymbol{t}) \exp(-2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{t}) \,\mathrm{d}\boldsymbol{t}$ is the $\boldsymbol{h}$-th Fourier coefficient of f.

Furthermore, $\rho_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) = \prod_{j=1}^{s} \rho_{\alpha,\gamma_j}(h_j)$, and

$$\rho_{\alpha,\gamma}(h) = \begin{cases} 1 & h = 0, \\ \gamma |h|^{-\alpha} & h \neq 0. \end{cases}$$

α is the “smoothness parameter”,

$1 = \gamma_1 \geq \gamma_2 \geq \ldots > 0$ are the coordinate weights.

Here: $\mathcal{P}_N = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is a lattice point set with generating vector $\boldsymbol{z} = (z_1, \ldots, z_s) \in \{1, \ldots, N-1\}^s$.

Points of $\mathcal{P}_N$:

$$\boldsymbol{x}_n = (x_{n,1}, \ldots, x_{n,s})$$

with

$$x_{n,j} = \left\{ \frac{n z_j}{N} \right\}.$$

For the Korobov space $\mathcal{H}_{s,\alpha,\gamma}$, and for a lattice point set $\mathcal{P}_N$, we have an explicit formula for $e^2(\mathcal{H}_{s,\alpha,\gamma}, \mathcal{P}_N)$.

$$e_N^2(\mathcal{H}_{s,\alpha,\gamma}, \mathcal{P}_N) = e_{N,s,\alpha,\boldsymbol{\gamma}}^2(\boldsymbol{z}) := \sum_{\boldsymbol{h}\in\mathcal{D}(\boldsymbol{z})\setminus\{0\}} \rho_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}),$$

where

$$\mathcal{D}(\boldsymbol{z}) := \{\boldsymbol{h} \in \mathbb{Z}^s \ : \ \boldsymbol{h}\cdot\boldsymbol{z} \equiv 0 \ (N)\}.$$

Finite formula:

$$e^2_{N,s,\alpha,\gamma}(\boldsymbol{z}) = -1 + \frac{1}{N}\sum_{n=0}^{N-1}\prod_{j=1}^{s}\left(1+\gamma_j\varphi_\alpha\left(\left\{\frac{nz_j}{N}\right\}\right)\right),$$

where $\varphi_\alpha\left(\frac{k}{N}\right)$ *can be precomputed for all values of* $k = 0, \ldots, N-1$.

If $\alpha = 2k$, $k \in \mathbb{N}$, φ_α is a constant multiple of the Bernoulli polynomial of degree α.

- All that remains is to find “good” $\boldsymbol{z} \in \{1, \ldots, N-1\}^s$.
- Rather big search space! (e.g., $N = 10\,000$ and $s = 20$).
- Component by component (CBC) construction: construct z_j one at a time.
  Size of search space is $N - 1$ per component.
- Can do fast CBC (Cools & Nuyens), computation cost of $\mathcal{O}(sN \log N)$.
- Computation cost of $\mathcal{O}(sN \log N)$ can still be demanding for big N, s
- Might want to have big N, s simultaneously.

Tractability

Let $e(N, s)$ be the Nth minimal (QMC) worst-case error,

$$e(N, s) = \inf_{\mathcal{P}} e_N(\mathcal{H}_{s,\alpha,\gamma}, \mathcal{P}),$$

where the infimum is extended over all N-element point sets $\mathcal{P}$ in $[0, 1]^s$.

Consider the (QMC) information complexity,

$$N_{\min}(\varepsilon, s) = \min\{N \in \mathbb{N} : e(N, s) \leq \varepsilon\}.$$

We say that integration in $\mathcal{H}_{s,\alpha,\gamma}$ is

- weakly QMC tractable, if

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_{\min}(\varepsilon, s)}{s+\varepsilon^{-1}} = 0;$$

- polynomially QMC-tractable, if there exist $c, p, q \geq 0$ such that

$$N_{\min}(\varepsilon, s) \leq c s^q \varepsilon^{-p}. \tag{1}$$

  Infima over all q and p such that (1) holds: s- and ε-exponent of polynomial tractability, respectively;

- strongly polynomially QMC-tractable, if (1) holds with $q = 0$. Infimum over all p such that (1) holds: ε-exponent of strong polynomial tractability.

For the Korobov space $\mathcal{H}_{s,\alpha,\gamma}$ it is known that:

- $$\sum_{j=1}^{\infty} \gamma_j < \infty$$
  is equivalent to strong polynomial tractability.
- If
  $$\sum_{j=1}^{\infty} \gamma_j^{1/\tau} < \infty$$
  for some $\tau \in [1, \alpha)$, then one can set the ε-exponent to $2/\tau$.
- The ε-exponent of $2/\alpha$ is optimal.
- Use CBC-constructed lattice point sets to obtain optimal results.

Suppose now that

$$\sum_{j=1}^{\infty} \gamma_j^{1/\tau} < \infty$$

for some $\tau > \alpha$.

So far: CBC construction of lattice point sets that yield optimal ε-exponent, but cost of CBC-construction is independent of the weights.

Our new result: CBC construction of lattice point sets that yield optimal ε-exponent, but cost of CBC-construction may decrease with the weights.
Exploit situations where weights decrease sufficiently fast.

The reduced CBC construction

Idea: make search space smaller for later components.

- Let N be a prime power, $N = b^m$, b prime, $m \in \mathbb{N}$
- Let $w_1, \ldots, w_s \in \mathbb{N}_0$ with $0 = w_1 \leq \ldots \leq w_s$
- Consider the sequence of reduced search spaces

$$\mathcal{Z}_{N,w_j} := \begin{cases} \{1 \leq z < b^{m-w_j} : \gcd(z,N) = 1\} & \text{if} \quad w_j < m \\ \{1\} & \text{if} \quad w_j \geq m \end{cases}$$

- Note that

$$|\mathcal{Z}_{N,w_j}| := \begin{cases} b^{m-w_j-1}(b-1) & \text{if} \quad w_j < m \\ 1 & \text{if} \quad w_j \geq m \end{cases}$$

- write $Y_j := b^{w_j}$

Algorithm (Reduced CBC construction)

Let N, $w_1, \ldots, w_s$, and $Y_1, \ldots, Y_s$ be as above. Construct $\boldsymbol{z} = (Y_1 z_1, \ldots, Y_s z_s)$ as follows.

- *Set $z_1 = 1$.*
- *For $d \leq s$ assume that $z_1, \ldots, z_{d-1}$ have already been found. Now choose $z_d \in \mathcal{Z}_{N,w_d}$ such that*

  $$e^2_{N,d,\alpha,\gamma}((Y_1 z_1, \ldots, Y_d z_d, Y_d z_d))$$

  is minimized as a function of z_d.
- *Increase d and repeat the second step until $(Y_1 z_1, \ldots, Y_s z_s)$ is found.*

Usual CBC construction: $w_j = 0$ and $Y_j = 1$ for all j.

Theorem

Let $\boldsymbol{z} = (Y_1 z_1, \ldots, Y_s z_s) \in \mathbb{Z}^s$ *be constructed according to the reduced CBC algorithm. Then for every* $d \leq s$ *it is true that*

$$e_{N,d,\alpha,\gamma}((Y_1 z_1, \ldots, Y_d z_d)) \leq \left(2 \prod_{j=1}^{d} \left(1 + \gamma_j^{\frac{1}{\alpha - 2\delta}} 2\zeta\left(\frac{\alpha}{\alpha - 2\delta}\right) b^{w_j} \right) \right)^{\alpha/2 - \delta} N^{-\alpha/2 + \delta}$$

for all $\delta \in \left(0, \frac{\alpha - 1}{2}\right]$, *where* ζ *is the Riemann zeta function.*

Let $\delta \in (0, \frac{\alpha-1}{2}]$ and let $\boldsymbol{z}$ be constructed according to the reduced CBC algorithm.

- If
$$\lim_{s\to\infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j b^{w_j} = 0,$$
then we have weak tractability.
- If
$$A := \limsup_{s\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{\frac{1}{\alpha-2\delta}} b^{w_j}}{\log s} < \infty,$$
then we have polynomial tractability with ε-exponent at most $\frac{2}{\alpha-2\delta}$ and s-exponent at most $2\zeta(\frac{\alpha}{\alpha-2\delta})A$.

- If
$$B := \sum_{j=1}^{\infty} \gamma_j^{\frac{1}{\alpha-2\delta}} b^{w_j} < \infty,$$
then we have strong polynomial tractability with ε-exponent at most $\frac{2}{\alpha-2\delta}$.

The reduced fast CBC construction

- The fast CBC construction (Nuyens/Cools) for the non-reduced case ($w_j = 0$) has a computation cost of $\mathcal{O}(sN \log N)$.
- The idea also works for the reduced case and yields reduced cost by exploiting additional structure of the case $w_j > 0$.
- Bonus: once $w_j \geq m$ the search space contains only one element. Thus the construction of additional components incurs no extra cost.
- The computational cost of the reduced fast CBC construction is

$$\mathcal{O}\left(N \log N + \min\{s, s^*\} N + \sum_{j=1}^{\min\{s,s^*\}} (m - w_j) N b^{-w_j} \right),$$

where $s^* := \min\{j \in \mathbb{N} : w_j \geq m\}$.

Example:

- Suppose weights γ_j are $\gamma_j = j^{-3}$.
- Fast CBC construction needs $\mathcal{O}(smb^m)$ operations to compute a generating vector for which the worst-case error is bounded independently of the dimension.
- Reduced fast CBC construction: choose, e.g., $w_j = \lfloor \frac{3}{2} \log_b j \rfloor$.
- We need $\mathcal{O}(mb^m + \min\{s, s^*\} mb^m)$ operations to compute a generating vector for which the worst-case error is still bounded independently of the dimension, as

$$\sum_j \gamma_j b^{w_j} < \zeta(3/2) < \infty.$$

- Reduced fast CBC construction significantly reduces computation cost.

Computation times and $\log_{10}$ worst case error for $b = 2$, $\alpha = 2$, $\gamma_j = j^{-3}$:

	$s = 10$	$s = 20$	$s = 50$
$m = 10$	0.384 -1.90	0.724 -1.88	1.80 -1.88
$m = 12$	1.32 -2.40	2.62 -2.37	6.55 -2.37
$m = 14$	5.22 -2.90	10.4 -2.87	26.0 -2.86
$m = 16$	21.7 -3.40	43.4 -3.36	109. -3.35

$w_j = 0$

	$s = 10$	$s = 20$	$s = 50$
$m = 10$	0.104 -1.89	0.120 -1.85	0.144 -1.79
$m = 12$	0.356 -2.39	0.400 -2.35	0.472 -2.31
$m = 14$	1.29 -2.88	1.45 -2.84	1.67 -2.79
$m = 16$	5.13 -3.39	5.68 -3.34	6.47 -3.30

$w_j = \lfloor \frac{3}{2} \log_b j \rfloor$

	$s = 10$	$s = 20$	$s = 50$	$s = 100$	$s = 200$	$s = 500$	$s = 1000$
$m = 10$	0.104 -1.89	0.120 -1.85	0.144 -1.79	0.148 -1.74	0.156 -1.67	0.164 -1.65	0.176 -1.65
$m = 12$	0.356 -2.39	0.400 -2.35	0.472 -2.31	0.524 -2.27	0.564 -2.19	0.588 -2.10	0.608 -2.08
$m = 14$	1.29 -2.88	1.45 -2.84	1.67 -2.79	1.88 -2.76	2.03 -2.72	2.35 -2.62	2.50 -2.53
$m = 16$	5.13 -3.39	5.68 -3.34	6.47 -3.30	7.16 -3.28	7.78 -3.24	9.27 -3.17	11.2 -3.10
$m = 18$	22.3 -3.89	24.4 -3.84	27.2 -3.81	29.4 -3.79	32.1 -3.76	38.2 -3.71	47.2 -3.65
$m = 20$	118. -4.41	126. -4.35	137. -4.33	145. -4.31	157. -4.30	182. -4.26	223. -4.21

Concluding remarks

- Reduced CBC constructions also works for general weights.
- Fast reduced CBC construction so far only for product weights.
- Everything (including fast construction for product weights) can be done analogously for a Walsh space with polynomial lattice points instead of lattice points.
- Instead of setting $z_j = 1$ if $w_j \geq m$, we can choose these z_j at random. Error bound essentially stays the same.
- If $w_j \geq m$, we can even replace the components of the lattice point set by uniformly distributed random points. We then have a hybrid point set in the sense of Spanier, the error bound stays the same.
- Error in Korobov space can be related to error of suitably transformed lattice points in Sobolev spaces.

Thank you very much for your attention.