High-dimensional and infinite-dimensional hyperbolic crosses and their applications in approximation and uncertainty quantification

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This talk is based in the recent joint works:

1 DD and T. Ullrich, N-Widths and ε-dimensions for high-dimensional approximations, Foundations Comp. Math. 13 (2013), 965-1003.

2 A. Chernov and DD, New explicit-in-dimension estimates for the cardinality of high-dimensional hyperbolic crosses and approximation of functions having mixed smoothness, (2014) http://arxiv.org/abs/1309.5170.

3 DD and M. Griebel, Hyperbolic cross approximation in infinite dimensions and applications in sPDEs, Manuscript (2014).

The “curse of dimensionality”

- There has been a great interest in solving numerical problems involving functions of big number s of variables.
- By classical methods as usually, the computation cost grows exponentially in s.
- We suffer the “curse of dimensionality” [Bellmann,1957] (“Dimensionality” is referred to the number s of variables).
- A way to get rid of it is
  - to assume that mixed derivatives of functions are bounded,
  - then to apply hyperbolic cross (HC) approximation.

Infinite-dimensional approximation

- The efficient approximation of a function of infinitely many variables is important in applications in physics, finance, engineering and statistics.
- It arises in UQ, computational finance and computational physics and is encountered for stochastic or parametric PDEs.
- Attempt: Apply infinite-dimensional HC to construct a linear approximation method to the solution of stochastic or parametric PDEs.

Classical hyperbolic crosses

- Classical HCs $\Gamma(s, T)$ are a domain of frequencies of trigonometric polynomials used for approximations of periodic functions having mixed derivative. They are given by

$$\Gamma(s, T) := \Big\{ \mathbf{k} \in \mathbb{Z}^s : \prod_{i=1}^{s} \max(|k_i|, 1) \le T \Big\}.$$

- Their cardinality is estimated as

$$C(s)\, T \log^{s-1} T \ \le\ |\Gamma(s, T)| \ \le\ C'(s)\, T \log^{s-1} T,$$

where $|G|$ denotes the cardinality of G.

n-Widths and ε-dimensions

- Kolmogorov n-widths:

$$d_n(W,H) := \inf_{\{L_n \text{ linear subspaces}, \ \dim L_n \le n\}} \sup_{f \in W} \inf_{g \in L_n} \|f - g\|_H.$$

- ε-dimension is the inverse of $d_n(W,H)$:

$$n_\varepsilon(W,H) := \inf\{n : \exists L_n : \dim L_n \le n, \sup_{f \in W} \inf_{g \in L_n} \|f - g\|_H \le \varepsilon\}.$$

- $n_\varepsilon(W,H)$ is the necessary dimension of linear subspace for approximation of functions from W with accuracy ε.

- From the computational view it is more convenient to study $n_\varepsilon(W,H)$ since it is directly related to the computation cost.

High-dimensional approach

- Sobolev space of mixed smoothness $\alpha \in \mathbb{N}$

$$\|f\|^2_{H^\alpha_{mix}} = \sum_{|\mathbf{k}|_\infty \le \alpha} \left\| \frac{\partial^{|\mathbf{k}|_1} f}{\partial x_1^{k_1} \cdots \partial x_s^{k_s}} \right\|_2^2, \quad |\mathbf{k}|_\infty := \max_{1 \le i \le s} k_i.$$

- U^α_{mix} is the unit ball in H^α_{mix}.
- Traditional estimations [Babenko 1960]:

$$A(\alpha, s)\, \varepsilon^{-1/\alpha}\, |\log \varepsilon|^{s-1} \le n_\varepsilon(U^\alpha_{mix}, L_2) \le A'(\alpha, s)\, \varepsilon^{-1/\alpha}\, |\log \varepsilon|^{s-1}.$$

- Our goal: to compute $A(\alpha, s)$, $A'(\alpha, s)$ explicitly in s.
- The basis for estimation of n_ε: Reduction to computation of cardinality of the associated HCs:

$$\boxed{|\Gamma(s, 1/\varepsilon)| - 1 \le n_\varepsilon(U^\alpha_{mix}, L_2) \le |\Gamma(s, 1/\varepsilon)|}$$

Plan of our talk

- High-dimensional HC approximation for two models of mixed smoothness:
  - Dyadic version [DD&Ullrich 2013];
  - Korobov version [Chernov&DD 2014].
- Infinite-dimensional HC approximation for two models of regularity:
  - Korobov version [DD&Griebel 2014];
  - Analytic version [DD&Griebel 2014].
- Application of infinite-dimensional HC approximation in stochastic or parametric PDEs.

Dyadic version: decomposition in frequency domain

- $L_2(\mathbb{T}^s)$ is the space of periodic functions on the torus $\mathbb{T}^s := [0,1]^s$ equipped with the inner product $(f,g) := \int_{\mathbb{T}^s} f(\mathbf{x})\overline{g(x)}\, d\mathbf{x}$.
- Let $e_{\mathbf{k}}(\mathbf{x}) := \prod_{j=1}^s e^{2\pi i\, k_j x_j}$.
- For $\mathbf{m} \in \mathbb{Z}^s_+$ and $f \in L_2(\mathbb{T}^s)$, define the operator:

  $\delta_{\mathbf{m}}(f) := \sum_{\mathbf{k}\in\square_{\mathbf{m}}} \hat{f}(\mathbf{k}) e_{\mathbf{k}}, \quad \square_{\mathbf{m}} := \{\mathbf{k} \in \mathbb{Z}^s : \lfloor 2^{m_i-1} \rfloor \le |k_i| < 2^{m_i}\},$

  where $\hat{f}(\mathbf{k})$ is the $\mathbf{k}$th Fourier coefficient.
- Based on Parseval's identity $\|f\|_2^2 = \sum_{\mathbf{m}\in\mathbb{Z}^s_+} \|\delta_{\mathbf{m}}(f)\|_2^2$, we define the space H^{α}_{mix} of mixed smoothness α:

$$\|f\|^2_{H^{\alpha}_{mix}} := \sum_{\mathbf{m}\in\mathbb{Z}^s_+} 2^{2\alpha|\mathbf{m}|_1} \|\delta_{\mathbf{m}}(f)\|_2^2 < \infty, \quad |\mathbf{m}|_1 := \sum_{j=0}^{s} m_j.$$

Dyadic version: Fourier HC approximation

- *Step HCs* are formed from dyadic boxes $\square_{\mathbf{m}}$:

$$G_{\text{step}}(s,n) := \Big\{ \bigcup \square_{\mathbf{m}} : \mathbf{m} \in \mathbb{Z}^s_+, \ |\mathbf{m}|_1 \le n \Big\}.$$

- $V^s(n)$ – the trigonometric polynomials with frequencies in $G_{\text{step}}(s,n)$.
- Linear Fourier operator:

$$S_n(f) := \sum_{\mathbf{k} \in G_{\text{step}}(s,n)} \hat{f}(\mathbf{k})\, e_{\mathbf{k}}.$$

- Let U^α_{mix} be the unit ball in H^α_{mix}. For $n \in \mathbb{N}$,

$$\sup_{f \in U^\alpha_{mix}} \ \inf_{g \in V^s(n)} \|f - g\|_2 \ \le \ \sup_{f \in U^\alpha_{mix}} \|f - S_n(f)\|_2 \ \le \ 2^{-n};$$

- We have $\dim V^s(n) = |G_{\text{step}}(s,n)|$.

Dyadic version: n_ε and the cardinality of HCs

- Estimation of n_ε is reduced to estimation of $|G_{\text{step}}(s,n)|$ for $\varepsilon = 2^{-n}$:

$$\boxed{|G_{\text{step}}(s,n)| - 1 \ \le\ n_\varepsilon(U^\alpha_{mix}, L_2(\mathbb{T}^s)) \ \le\ |G_{\text{step}}(s,n)|,}$$

- For any $n \in \mathbb{Z}_+$,

$$2^n \binom{n+s-1}{s-1} \ \le\ |G_{\text{step}}(s,n)| \le 2^{n+1} \binom{n+s-1}{s-1}.$$

Theorem (DD&Ullrich 2014)

Let $\alpha > 0$. *Then we have for any* $0 < \varepsilon \le 2^{-\alpha s}$,

$$\frac{1}{2}[\alpha(s-1)]^{-(s-1)} \le \frac{n_\varepsilon(U^\alpha_{mix}, L_2(\mathbb{T}^s))}{\varepsilon^{-1/\alpha}|\log \varepsilon|^{s-1}} \le 4\left(\frac{\alpha(s-1)}{2e}\right)^{-(s-1)}.$$

- The ratio decays exponentially fast in s.

Dyadic version

- [DD&Ullrich 2013] Estimates in this manner have been proven also for $n_\varepsilon(U^\alpha_{mix}, H^\gamma(\mathbb{T}^s))$ in energy norm of Sobolev space $H^\gamma(\mathbb{T}^s)$.

- In the dyadic version, we can prove lower and upper bounds for $n_\varepsilon(U^\alpha_{mix}, L_2(\mathbb{T}^s))$ only for very small $\varepsilon \le 2^{-\alpha s}$.

- The reason: The step HC approximation for the class U^α_{mix} involve a whole dyadic block

$$\delta_{\mathbf{k}}(f) := \sum_{\mathbf{m} \in \square_{\mathbf{k}}} \hat{f}(\mathbf{m}) e_{\mathbf{m}}$$

with the cardinality $|\square_{\mathbf{k}}| \ge 2^s$.

- Let us consider another model of mixed smoothness: Korobov-type mixed smoothness.

A modification of Korobov space K_s^r

- For $r > 0$ and $\mathbf{k} \in \mathbb{Z}^s$, define the scalar $\lambda(\mathbf{k})$ by

$$\lambda(\mathbf{k}) := \prod_{j=1}^{s} \lambda(k_j), \quad \lambda(k_j) := \ (1 + |k_j|),$$

- Korobov function:

$$\kappa_s^r \ := \ \sum_{\mathbf{k} \in \mathbb{Z}^s} \lambda(\mathbf{k})^{-r} \, e_{\mathbf{k}}.$$

- Korobov space K_s^r:

$$K_s^r := \{f : f = \kappa_s^r * g, \ g \in L_2(\mathbb{T}^s)\}$$

with the norm

$$\|f\|_{K_s^r} \ := \|g\|_2.$$

Hyperbolic crosses for K_s^r

- The symmetric continuous HC:

$$G(s,T) := \left\{ \mathbf{k} \in \mathbb{Z}^s : \prod_{i=1}^{s} (|k_i| + 1) \le T \right\}.$$

- U_s^r is the unit ball in K_s^r.
- Using Fourier approximation by trigonometric polynomials with frequencies in HC $G(s,T)$ we have

$$\boxed{|G(s,T)| - 1 \le n_\varepsilon(U_s^r, L_2(\mathbb{T}^s)) \le |G(s,T)|, \ \ T = \varepsilon^{-1/r}.}$$

- $\Rightarrow$ Estimation of $n_\varepsilon(U_s^r, L_2(\mathbb{T}^s))$ is reduced to estimation of $|G(s,T)|$.

New estimates for the cardinality of HCs

Theorem (Chernov&DD 2014)

For $T \geq 1$,

$$|G(s,T)| \; < \; \frac{2^s \, T(\ln T + s \ln 2)^s}{(s-1)! \, \big(\ln T + s\ln 2 + s - 1\big)},$$

and for $T > (3/2)^s$,

$$|G(s,T)| \; > \; \frac{2^s \, T(\ln T - s \ln(3/2))^s}{(s-1)! \, (\ln T - s\ln(3/2) + s)}$$

- For $\varepsilon > 0$,

$$\boxed{|G(s,T)| - 1 \leq n_\varepsilon(U^r_s, L_2(\mathbb{T}^s)) \leq |G(s,T)|, \;\; T = \varepsilon^{-1/r}.}$$

New bounds for $n_\varepsilon(U^r_s, L_2(\mathbb{T}^s))$

$\Rightarrow$

Theorem (Chernov&DD 2014)

- *Let* $r > 0$, $s \ge 2$. *Then we have for every* $\varepsilon \in (0,1]$,

$$n_\varepsilon(U^r_s, L_2(\mathbb{T}^s)) \ \le \ \frac{2^s\, \varepsilon^{-1/r}\, (\ln \varepsilon^{-1/r} + s \ln 2)^s}{(s-1)!\, \big(\ln \varepsilon^{-1/r} + s \ln 2 + s - 1 \big)},$$

and for every $\varepsilon \in (0, [3/2]^{-sr})$,

$$n_\varepsilon(U^r_s, L_2(\mathbb{T}^s)) \ \ge \ \frac{2^s\, \varepsilon^{-1/r}\, (\ln \varepsilon^{-1/r} - s \ln(3/2))^s}{(s-1)!\, (\ln \varepsilon^{-1/r} - s \ln(3/2) + s)} \ - \ 1$$

- In traditional estimations, $\varepsilon^{-1/r}|\log \varepsilon|^{(s-1)/r}$ is a priori split from constants which are a function of dimension parameter s. $\Rightarrow$ Any high-dimensional estimate based on them leads to a rougher bound.

Related results

- [Kühn, Sickel & Ullrich 2014] have established upper and lower bounds explicit in s for large n and small n (preasymptotics), for the approximation number

$$a_n(I_s : H^\alpha_{\mathrm{mix}} \to L_2(\mathbb{T}^s))$$

(given in the talk by Winfried Sickel yesterday).

Infinite tensor product probability measure

- Let $\mathbb{I} := [-1,1]$. Let $\mathrm{d}\mu_\infty$ be the tensor product infinite tensor product measure on $\mathbb{I}^\infty$ of the univariate uniform probability measures on $\mathbb{I}$:
$$d\mu_\infty(\mathbf{y}) := \bigotimes_{j\in\mathbb{Z}} \frac{1}{2} dy_j.$$
The sigma algebra Σ for $\mathrm{d}\mu_\infty$ is generated by the finite rectangles
$$\prod_{j\in\mathbb{N}} I_j, \quad I_j \subset \mathbb{I},$$
where only a finite number of the I_j are different from $\mathbb{I}$.

- Let $\mathrm{d}\mu_n = \mathrm{d}\mathbf{x}$ be the uniform probability measure on the n-dimensional torus $\mathbb{T}^n := [0,1]^n$. We define
$$d\mu(\mathbf{x},\mathbf{y}) := d\mu_n(\mathbf{x}) \bigotimes d\mu_\infty(\mathbf{y}).$$

Space $L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$

- $L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$ is the Hilbert space of functions on $\mathbb{T}^n \otimes \mathbb{I}^\infty$:

$$(f, g) := \int_{\mathbb{I}^\infty} f(\mathbf{x}, \mathbf{y})\overline{g(\mathbf{x}, \mathbf{y})}\, d\mu(\mathbf{x}, \mathbf{y}).$$

- The norm in $L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$ is defined as $\|f\| := (f, f)^{1/2}$.
- $$L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu) \;=\; L_2(\mathbb{T}^n, d\mu_n) \otimes L_2(\mathbb{I}^\infty, d\mu_\infty).$$

Bochner space $\mathcal{H} := H^b(\mathbb{T}^n) \otimes L_2(\mathbb{I}^\infty, d\mu_\infty)$

- For $b \ge 0$,

  $\mathcal{H} := H^b(\mathbb{T}^n) \otimes L_2(\mathbb{I}^\infty, d\mu_\infty)$,

  $H^b(\mathbb{T}^n)$ is the Sobolev space of smoothness b.

- $\Rightarrow$ $\mathcal{H}$ is a Bochner space:

  $\mathcal{H} = L_2(\mathbb{I}^\infty, d\mu_\infty; H^b(\mathbb{T}^n))$

  – the set of all functions $f : \mathbb{I}^\infty \to H^b(\mathbb{T}^n)$ such that

$$\|f\|_{\mathcal{H}}^2 := \int_{\mathbb{I}^\infty} \|f(\cdot, \mathbf{y}\|_{H^b(\mathbb{T}^n)}^2 \, d\mu_\infty(\mathbf{y}) \ < \ \infty.$$

- For $b = 0$,

$$\mathcal{H} \ = \ L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu) \ = \ L_2(\mathbb{T}^n, d\mu_n) \otimes L_2(\mathbb{I}^\infty, d\mu_\infty).$$

Infinite-dimensional Legendre polynomials

- Let $\{L_k\}_{k=0}^{\infty}$ be the family of univariate orthonormal Legendre polynomials in $L_2(\mathbb{I}, \frac{1}{2}dx)$, i.e.

$$\frac{1}{2}\int_{\mathbb{I}} L_k(y)L_s(y)dy = \delta_{ks}.$$

- $\mathbb{Z}^\infty$ – the set of all sequences $\mathbf{k} = (k_j)_{j=1}^{\infty}$ with $k_j \in \mathbb{Z}$;

  $\mathbb{Z}^\infty_{+*} := \{\mathbf{k} \in \mathbb{Z}^\infty : k_j \geq 0,\ j = 1, 2, ..., \operatorname{supp}(\mathbf{k}) \text{ is finite}\}.$

- For $\mathbf{s} \in \mathbb{Z}^\infty_{+*}$, we define

$$L_{\mathbf{s}}(\mathbf{y}) := \prod_{j \in \operatorname{supp}(\mathbf{s})} L_{s_j}(y_j).$$

- $(L_{\mathbf{s}})_{\mathbf{s}\in\mathbb{Z}^\infty_{+*}}$ is an orthonormal basic of $L_2(\mathbb{I}^\infty, d\mu)$.

Infinite-dimensional mixed polynomials

- Let $\{e_{\mathbf{k}}\}_{k \in \mathbb{Z}^n}$ be the orthonormal trigonometric basis in $L_2(\mathbb{T}^n, d\mathbf{x})$.
- For $(\mathbf{k}, \mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}^{\infty}_{+*}$, we define

$$h_{(\mathbf{k},\mathbf{s})}(\mathbf{x}, \mathbf{y}) := \ e_{\mathbf{k}} \, L_{\mathbf{s}}.$$

- $(h_{(\mathbf{k},\mathbf{s})})_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}^{\infty}_{+*}}$ is an orthonormal basis of $L_2(\mathbb{T}^n \otimes \mathbb{I}^{\infty}, d\mu)$.
- For every $f \in L_2(\mathbb{T}^n \otimes \mathbb{I}^{\infty}, d\mu)$,

$$f = \sum_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}^{\infty}_{+*}} f_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}, \quad f_{(\mathbf{k},\mathbf{s})} = (f, h_{(\mathbf{k},\mathbf{s})}).$$

Infinite-variate Korobov-type space

- Let $n \in \mathbb{Z}_+$, $a > 0$ and $\mathbf{r} = (r_j)_{j=1}^\infty \in \mathbb{I}^\infty$ with $r_j > 0$.

  For a $(\mathbf{k}, \mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}_{+*}^\infty$, we define

$$\lambda_{a,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) := \max_{1 \le j \le n} (1 + |k_j|)^a \prod_{j=1}^\infty (1 + s_j)^{r_j}.$$

- The Korobov-type space $K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ is the set of all functions $f \in L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$ such that

$$f = \sum_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}_{+*}^\infty} \lambda_{a,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) g_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}, \quad g \in L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu).$$

- The norm of $K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ is defined by

$$\|f\|_{K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)} := \|g\|.$$

Infinite-variate Korobov-type space

- For $n = 0$,
$$K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty) := \ K^{\mathbf{r}}(\mathbb{I}^\infty).$$

- $$K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty) \ = \ H^a(\mathbb{T}^n) \otimes K^{\mathbf{r}}(\mathbb{I}^\infty).$$

- The subspace $K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^s)$ in $K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ is the set of all functions $f \in L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$ such that

$$f = \sum_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}^\infty_{+*}: \ \mathrm{supp}(\mathbf{k}) \subset \{1,\cdots,s\}} \lambda_{a,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) g_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})},$$

with $g \in L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$.

Infinite-dimensional HC approximation

- The infinite-dimensional HC with different weights:

$$G(T) := \left\{(\mathbf{k}, \mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}^\infty_{+*} : \lambda_{a-b,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) \leq T\right\} \ (a > b \geq 0).$$

- Let $\mathcal{P}(T)$ be the HC subspace of polynomials g of the form

$$g = \sum_{(\mathbf{k},\mathbf{s}) \in G(T)} g_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}.$$

- $\dim \mathcal{P}(T) = |G(T)|$.

- The HC operator $S_T : \mathcal{H} \to \mathcal{P}(T)$

$$S_T(f) := \sum_{(\mathbf{k},\mathbf{s}) \in G(T)} f_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}.$$

Infinite-dimensional HC approximation

- $\mathcal{H} := H^b(\mathbb{T}^n) \otimes L_2(\mathbb{I}^\infty, d\mu_\infty)$;
  $U^{a,\mathbf{r}}$ is the unit ball in $K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$;
  $U^{a,\mathbf{r}}_s$ is the unit ball in $K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^s)$.

- For arbitrary $T \geq 1$,

$$\boxed{\sup_{f \in U^{a,\mathbf{r}}} \inf_{g \in \mathcal{P}(T)} \|f - g\|_{\mathcal{H}} = \sup_{f \in U^{a,\mathbf{r}}} \|f - S_T(f)\|_{\mathcal{H}} \leq T^{-1}}$$

- For $\varepsilon \in (0, 1]$,

$$\boxed{|H(1/\varepsilon)| - 1 \ \leq \ n_\varepsilon(U^{a,\mathbf{r}}, \mathcal{H})) \ \leq \ |H(1/\varepsilon)|}$$

Cardinality of infinite-dimensional HCs

Theorem (DD&Griebel 2014)

Let $n \in \mathbb{N}$, $0 \le b < a < r/n$ *and* $\mathbf{r} = (r_j)_{j=1}^\infty \in \mathbb{I}^\infty$ *with*

$$0 < r = r_{n+1} = \cdots = r_{n+\nu+1} < r_{n+\nu+2} \le r_{n+\nu+3} \le \cdots.$$

Assume that there holds the condition

$$M := \sum_{j=\nu+2}^{\infty} \frac{1}{nr_j/(a-b)-1} \left(\frac{3}{2}\right)^{-(nr_j/(a-b)-1)} \quad < \quad \infty.$$

Then we have for every $T \ge 1$,

$$\boxed{\lfloor T^{n/(a-b)} \rfloor \ \le \ |G(T)| \ \le \ C(a,n,\mathbf{r})\, T^{n/(a-b)},} \tag{1}$$

where

$$C(a,b,n,\mathbf{r}) := \ e^M\, 3^n \ \left(1 + \tfrac{1}{rn/(a-b)-1} \left(\tfrac{3}{2}\right)^{rn/(a-b)-1}\right)^{\nu}.$$

Estimates of ε-dimensions

Theorem (DD&Griebel 2014)

Let

$$n_\varepsilon := n_\varepsilon(U^{a,\mathbf{r}}, \mathcal{H}),$$

$$n_\varepsilon(s) := n_\varepsilon(U^{a,\mathbf{r}}_s, \mathcal{H})).$$

Under the assumptions and notation of the previous theorem, we have for every $\varepsilon \in (0,1]$,

$$\boxed{\lfloor \varepsilon^{-n/(a-b)} \rfloor - 1 \ \le \ n_\varepsilon(s) \ \le \ n_\varepsilon \ \le \ C(a,n,\mathbf{r})\, \varepsilon^{-n/(a-b)}.}$$

Remarks

- The terms $C(a, b, n, \mathbf{r})$ is independent of s when s may be very large.
- The main component $\varepsilon^{-n/(a-b)}$ depends on ε and a, b, n, only.
- The restriction on $\mathbf{r}$

$$\sum_{j=\nu+2}^{\infty} \frac{1}{n\,r_j/(a-b)-1}\left(\frac{3}{2}\right)^{-(n\,r_j/(a-b)-1)} < \infty$$

  is moderate. It is satisfied if a, b, n are fixed and the subsequence $(r_j)_{j=\nu+2}^{\infty}$ is mildly increasing say as an arithmetic progression.
- The problem of $n_\varepsilon(U_s^{a,\mathbf{r}}, L_2(\mathbb{T}^n \otimes \mathbb{R}^s, d\mu))$ is strongly polynomially tractable with respect to large s.

Infinite-variate space $A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$

- Let $n \in \mathbb{Z}_+$, $a > 0$ and $\mathbf{r} = (r_j)_{j=1}^\infty \in \mathbb{I}^\infty$ with $r_j > 0$. For a $(\mathbf{k}, \mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}_{+*}^\infty$, we define

$$\rho_{a,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) := \max_{1 \le j \le n} (1 + |k_j|)^a \exp(\mathbf{r}, \mathbf{s}), \quad (\mathbf{r}, \mathbf{s}) := \sum_{j=1}^\infty r_j s_j.$$

- The space $A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ is the set of all functions $f \in L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu)$ such that

$$f = \sum_{(\mathbf{k},\mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}_{+*}^\infty} \rho_{a,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) g_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}, \quad g \in L_2(\mathbb{T}^n \otimes \mathbb{I}^\infty, d\mu).$$

- The norm of $A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ is defined by

$$\|f\|_{A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)} := \|g\|.$$

Infinite-variate space $A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty) \ = \ H^a(\mathbb{T}^n) \otimes A^{\mathbf{r}}(\mathbb{I}^\infty)$

- For $n = 0$,
$$K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty) := \ A^{\mathbf{r}}(\mathbb{I}^\infty).$$
-
$$A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty) \ = \ H^a(\mathbb{T}^n) \otimes A^{\mathbf{r}}(\mathbb{I}^\infty).$$
- $\Rightarrow \ A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ is a subspace of $\mathcal{H}$ for $a > b$.

HC approximation in Bochner space

- The infinite-dimensional exp. HC with different weights:

$$E(T) := \left\{(\mathbf{k}, \mathbf{s}) \in \mathbb{Z}^n \otimes \mathbb{Z}^\infty_{+*} : \rho_{a-b,n,\mathbf{r}}(\mathbf{k}, \mathbf{s}) \le T\right\}, \ (a > b).$$

- Let $\mathcal{E}(T)$ be the HC subspace of polynomials g of the form

$$g = \sum_{(\mathbf{k},\mathbf{s}) \in E(T)} g_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}.$$

- The HC operator:

$$P_T(f) := \sum_{(\mathbf{k},\mathbf{s}) \in E(T)} f_{(\mathbf{k},\mathbf{s})} h_{(\mathbf{k},\mathbf{s})}.$$

HC approximation in Bochner space

- $\mathcal{H} := \; H^b(\mathbb{T}^n) \otimes L_2(\mathbb{I}^\infty, d\mu_\infty)$.
- $B^{a,\mathbf{r}}$ is the unit ball in $A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty) \; = \; H^a(\mathbb{T}^n) \otimes A^{\mathbf{r}}(\mathbb{I}^\infty)$.
- Let $a > b \geq 0$. For arbitrary $T \geq 1$,

$$\boxed{\sup_{f \in B^{a,\mathbf{r}}} \inf_{g \in \mathcal{E}(T)} \|f - g\|_{\mathcal{H}} = \sup_{f \in B^{a,\mathbf{r}}} \|f - P_T(f)\|_{\mathcal{H}} \leq T^{-1}.}$$

- For $\varepsilon \in (0, 1]$,

$$\boxed{|E(1/\varepsilon)| - 1 \; \leq \; n_\varepsilon(B^{a,\mathbf{r}}, \mathcal{H}) \; \leq \; |E(1/\varepsilon)|.}$$

Cardinality of infinite-dimensional HCs

Theorem (DD&Griebel 2014)

Let $n \in \mathbb{N}$, $a > b \geq 0$ *and* $\mathbf{r} = (r_j)_{j=1}^\infty \in \mathbb{I}^\infty$ *with* $r_j > 0$. *Assume that there holds the condition*

$$\sum_{j=1}^\infty (e^{nr_j/(a-b)} - 1)^{-1} \ < \ \infty.$$

Then we have for every $T \geq 1$,

$$\boxed{\lfloor T^{n/(a-b)} \rfloor \ \leq \ |E(T)| \ \leq \ D(a,b,n,\mathbf{r})\, T^{n/(a-b)},} \tag{2}$$

where

$$D(a,b,n,\mathbf{r}) := \ 3^{2n} \exp\left[\sum_{j=1}^\infty (e^{nr_j/(a-b)} - 1)^{-1}\right].$$

ε-dimensions in Bochner space

Theorem (DD&Griebel 2014)

Under the assumptions and notation of the previous theorem, we have for every $\varepsilon \in (0,1]$,

$$\boxed{\lfloor \varepsilon^{-n/(a-b)} \rfloor - 1 \ \leq \ n_\varepsilon(B^{a,\mathbf{r}}, \mathcal{H})) \ \leq \ D(a,b,n,\mathbf{r})\, \varepsilon^{-n/(a-b)}.}$$

Related papers: Liberating the dimension for function approximation [Wasilkowski,Wozniakowski 2011], [Wasilkowski 2012].

Application in Elliptic sPDEs (periodic model)

- Let $D := (0,1)^n \subset \mathbb{R}^n$. Consider the following parametric (stochastic) periodic elliptic problem

$$-\nabla_{\mathbf{x}} (a(\mathbf{x},\mathbf{y})\nabla_{\mathbf{x}} u(\mathbf{x},\mathbf{y})) = f(\mathbf{x},\mathbf{y}) \text{ in } D, \quad u|_{\partial D} = 0, \qquad (3)$$

  where the diffusion coefficients $a(\mathbf{x},\mathbf{y})$ are functions 1-periodic functions in $\mathbf{x}$, and parameters $\mathbf{y} = (y_j)_{j=1}^\infty \in Y := \mathbb{I}^\infty$, and $f(\cdot,\mathbf{y})$ is a fixed 1-periodic function in $L_2(D)$.
- In a typical case, $a(\mathbf{x},\mathbf{y})$ has the following expansion

$$a(\mathbf{x},\mathbf{y}) := \bar{a}(\mathbf{x}) + \sum_{j=1}^\infty y_j \psi_j(\mathbf{x},\mathbf{y}), \qquad (4)$$

  where $\bar{a}$ is 1-periodic, $\bar{a} \in L_\infty(D)$ and $(\psi_j)_{j=1}^\infty \subset L_\infty(D)$ with 1-periodic ψ_j.
- A choice for $(\psi_j)_{j=1}^\infty$ in sPDEs is the Karhúnen-Loève basis where $\bar{a}$ is the average of a and y_j is pairwise decorrelated random variables.

Papers on sPDEs

- [Nobile,Tempone,Webster 2008] Sparse grid collcation method.
- [Cohen,DeVore,Schwab 2010], [Hoang,Schwab 2014] N-term Galerkin approximations.
- [Beck,Nobile,Tammelli,Tempone 2012, 2013] Polynomial approximation by Galerkin and collocation methods.

Application in Elliptic sPDEs (periodic model)

- The solution u is living in the Bochner spase

$$\mathcal{H}_0 := L_2(\mathbb{I}^\infty, d\mu; V), \quad V := H_0^b(\mathbb{T}^n) \ (b = 0, 1).$$

- Depending on the properties of the diffusion function $a(\mathbf{x}, \mathbf{y})$ and the function $f(\mathbf{x}, \mathbf{y})$, we have higher regularity of u in both, $\mathbf{x}$ and $\mathbf{y}$.

- We may assume that

  (K) $u \in H^a(\mathbb{T}^n) \otimes K^{\mathbf{r}}(\mathbb{I}^\infty) = K^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ (mixed smoothness), or

  (A) $u \in H^a(\mathbb{T}^n) \otimes A^{\mathbf{r}}(\mathbb{I}^\infty) = A^{a,\mathbf{r}}(\mathbb{T}^n \otimes \mathbb{I}^\infty)$ (analyticity)

  for some $a > b$ $(a = 1, 2)$, $\mathbf{r} \in \mathbb{R}_+^\infty$ satisfying the assumptions of the above theorems on the infinite-dimensional HCs $G(T)$ and $E(T)$.

- With some natural restrictions
  (for instance, $\|\psi_j\|_{L_\infty(D)} \le C j^{-s}$, $s > 1$, in K-L expansion),
  u has analytic regularity in $\mathbf{y}$ [Cohen,DeVore,Schwab 2010]
  $\Rightarrow$ the assumption (A) is quite reasonable.

Linear approximation to the solution u of elliptic sPDEs

- Let $a > b \geq 0$. For arbitrary $T \geq 1$,

$$\text{(K)} \quad \boxed{\|u - S_T(u)\|_{L_2(\mathbb{I}^\infty, d\mu; V)} \leq \|u\|_{H^a(\mathbb{T}^n) \otimes K^{\mathbf{r}}(\mathbb{I}^\infty)} T^{-1}}$$

and

$$\text{(A)} \quad \boxed{\|u - P_T(u)\|_{L_2(\mathbb{I}^\infty, d\mu; V)} \leq \|u\|_{H^a(\mathbb{T}^n) \otimes A^{\mathbf{r}}(\mathbb{I}^\infty)} T^{-1}}$$

- The cardinality index sets of the associated HCs are bounded by

$$\text{(K)} \quad |G(T)| \leq C(a, b, n, \mathbf{r}) T^{n/(a-b)},$$

and

$$\text{(A)} \quad |E(T)| \leq D(a, b, n, \mathbf{r}) T^{n/(a-b)}.$$

Linear approximation to the solution u of elliptic sPDEs

Case (K)

- $N := |G(T)| \ \Rightarrow \text{rank}\, S_T = N \Rightarrow S_T$ as a linear operator of rank N: $L_N := S_T$, for which

$$\boxed{\|u - L_N(u)\|_{L_2(\mathbb{I}^\infty, d\mu; V)} \ \le \ C^* \, \|u\|_{H^a(\mathbb{T}^n) \otimes K^{\mathbf{r}}(\mathbb{I}^\infty)} \, N^{-(a-b)/n}}$$

where $C^* := C(a, b, n, \mathbf{r})^{(a-b)/n}$.

Case (A)

- $N := |E(T)| \ \Rightarrow \text{rank}\, P_T = N \Rightarrow P_T$ as a linear operator of rank N: $\Lambda_N := P_T$, for which

$$\boxed{\|u - \Lambda_N(u)\|_{L_2(\mathbb{I}^\infty, d\mu; V)} \ \le \ D^* \, \|u\|_{H^a(\mathbb{T}^n) \otimes A^{\mathbf{r}}(\mathbb{I}^\infty)} \, N^{-(a-b)/n}}$$

where $D^* := D(a, b, n, \mathbf{r})^{(a-b)/n}$.

Conclusion

- We have shown that, under our assumptions and for linear information, the bounds are completely free of the dimension.
- In any case, the stochastic part has disappeared from the complexities and only appears in the constants.
- The analysis for standard information still needs to be done.

Thank you for your attention!