Preasymptotic estimates for approximation of multivariate Sobolev functions

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Approximation numbers

- **Approximation numbers**
  of bounded linear operators $T : X \to Y$ between two Banach spaces

$$a_n(T : X \to Y) := \inf\{\|T - A\| : \mathrm{rank}\, A < n\}$$

- **Interpretation in IBC and Numerical Analysis**
  Every operator $A : X \to Y$ of finite rank k can be written as

$$Ax = \sum_{j=1}^{k} L_j(x)\, y_j \qquad \text{for all } x \in X$$

with linear functionals $L_j \in X^*$ and vectors $y_j \in Y$.

$\curvearrowright$ A is a linear algorithm using k arbitrary linear informations

$$\|T - A\| = \sup_{\|x\| \le 1} \|Tx - Ax\| = \text{worst-case error of } A.$$

- For compact operators between Hilbert spaces one has
  $$a_n(T) = s_n(T) = n\text{-th singular number of } T\,.$$
- General problem in functional analysis or approximation theory:
  Find the asymptotic behaviour of $a_n(T)$ as $n \to \infty$.
  Typical results are of the form
  $$c\, n^{-\alpha} \le a_n(T) \le C\, n^{-\alpha} \text{ for all (or for large) } n \in \mathbb{N},$$
  with certain (often unspecified) constants.
- More relevant for practical issues, for instance in
  – tractability problems in IBC
  – error analysis of numerical algorithms
  is the preasymptotic behaviour of $a_n(T)$ i.e. estimates for small n
- **Our aims.** It is well known that $a_n(I_d : H^s(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim n^{-s/d}$.
  We will give
  – explicit constants, in particular asymptotic constants
  – sharp preasymptotic estimates in the range $2 \le n \le 2^d$
  with special emphasis on the dependence
  on the dimension d and on the chosen norm.

Isotropic Sobolev spaces – integer smoothness

- $\mathbb{T}^d$ is the d-dimensional torus
  $= [0, 2\pi]^d$ with identification of opposite points,
  equipped with the normalized Lebesgue measure $(2\pi)^{-d} dx$.
- **Sobolev spaces** on $\mathbb{T}^d$ of integer smoothness $m \in \mathbb{N}$

  $H^m(\mathbb{T}^d)$ consists of all $f \in L_2(\mathbb{T}^d)$ such that

  $$D^\alpha f \in L_2(\mathbb{T}^d) \quad \text{for all multi-indices } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq m\,.$$

- **Natural norm** (all partial derivatives)

  $$\| f \,| H^m(\mathbb{T}^d) \| := \Big(\sum_{|\alpha| \leq m} \| D^\alpha f \,| L_2(\mathbb{T}^d) \|^2 \Big)^{1/2}$$

- **Modified natural norm** (only highest derivatives in each coordinate)

  $$\| f \,| H^m(\mathbb{T}^d) \|^* := \Big(\| f \,| L_2(\mathbb{T}^d) \|^2 + \sum_{j=1}^d \Big\| \frac{\partial^m f}{\partial x_j^m} \Big| L_2(\mathbb{T}^d) \Big\|^2 \Big)^{1/2}$$

Fourier coefficients – equivalent norms

- **Fourier coefficients** of $f \in L_2(\mathbb{T}^d)$

$$c_k(f) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ikx} dx \quad , \quad k \in \mathbb{Z}^d$$

- Parseval's identity and $c_k(D^\alpha f) = (ik)^\alpha c_k(f)$
  $\Longrightarrow$ norms in $H^m(\mathbb{T}^d)$ can be expressed in terms of $c_k(f)$

- For the natural norm one has **equivalence**

$$\| f \,|H^m(\mathbb{T}^d)\| \sim \left(\sum_{k \in \mathbb{Z}^d} \Big(1 + \sum_{j=1}^d |k_j|^2 \Big)^m |c_k(f)|^2 \right)^{1/2}$$

with equivalence constants independent on d.

- For the modified natural norm one has even **equality**

$$\| f \,|H^m(\mathbb{T}^d)\|^* = \left(\sum_{k \in \mathbb{Z}^d} \Big(1 + \sum_{j=1}^d |k_j|^{2m} \Big) |c_k(f)|^2 \right)^{1/2} .$$

Fractional smoothness $s > 0$

- **Idea.** Replace, in der Fourier norm, $m \in \mathbb{N}$ with a real number $s > 0$.
  $\curvearrowright$ all norms are weighted ℓ_2-sums of Fourier coefficients
- $H^{s,p}(\mathbb{T}^d)$ consists of all $f \in L_2(\mathbb{T}^d)$ such that

$$\|f|H^{s,p}(\mathbb{T}^d)\| := \Big(\sum_{k \in \mathbb{Z}^d} w_{s,p}(k)^2 |c_k(f)|^2 \Big)^{1/2} < \infty \,,$$

where the weights are

$$w_{s,p}(k) = \begin{cases} (1 + |k_1|^p + \ldots + |k_d|^p)^{s/p} & , \ 0 < p < \infty \\ \max(1, |k_1|, \ldots, |k_d|)^s & , \ p = \infty \end{cases}$$

- For fixed $s > 0$ and $d \in \mathbb{N}$, all these norms are equivalent. The equivalence constants depend heavily on d, but clearly all spaces $H^{s,p}(\mathbb{T}^d)$, $0 < p \leq \infty$, coincide as vector spaces.

$p = 2 \iff$ **natural norm**

$p = 2s \iff$ **modified natural norm**

New results

- Existence and computation of the limits

$$\lim_{n\to\infty} n^{s/d}\, a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))$$

  for all $s > 0$, $d \in \mathbb{N}$ and $0 < p \leq \infty$
- Asymptotic behaviour of the constants as $d \to \infty$
- Explicit two-sided estimates of a_n for large n / small n
- Similar results for
  – approximation in the sup-norm
  – spaces of dominating mixed smoothness
  $\longrightarrow$ talk by Winfried Sickel

Reduction to sequence spaces

- **Commutative diagram**

$$
\begin{array}{ccc}
H^{s,p}(\mathbb{T}^d) & \xrightarrow{\quad I_d \quad} & L_2(\mathbb{T}^d) \\
\Big\downarrow A & & \Big\uparrow B \\
\ell_2(\mathbb{Z}^d) & \xrightarrow{\quad D \quad} & \ell_2(\mathbb{Z}^d)
\end{array}
$$

with $Af := (w_{s,p}(k)\, c_k(f))_{k\in\mathbb{Z}^d}$, $B\xi := \sum_{k\in\mathbb{Z}^d} \xi_k\, e^{ikx}$

and a diagonal operator $D(\xi_k) := (\xi_k / w_{s,p}(k))$

A and B are unitary operators $\Longrightarrow$ $a_n(I_d) = a_n(D) = s_n(D)$

Diagonal operators and combinatorics

- Let $(\sigma_n)_{n\in\mathbb{N}}$ be the non-increasing rearrangement of $(1/w_{s,p}(k))_{k\in\mathbb{Z}^d}$. With this piecewise constant sequence we have

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = s_n(D : \ell_2(\mathbb{Z}^d) \to \ell_2(\mathbb{Z}^d)) = \sigma_n \,.$$

- The "sequence" $(w_{s,p}(k))_{k\in\mathbb{Z}^d}$ attains all values $(1+r^p)^{s/p}$, $r\in\mathbb{N}$, in fact each of them at least $2d$ times, for $k=\pm re_1, \pm re_2 \ldots, \pm re_d$.

Lemma

Let $r\in\mathbb{N}$ and $n = \#\{k\in\mathbb{Z}^d : \sum_{j=1}^d |k_j|^p \le r^p\}$. Then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = \sigma_n = (1+r^p)^{-s/p} \,.$$

- In principle, this gives $a_n(I_d)$ for sufficiently many $n's$, but to compute these cardinalities exactly is impossible. However, good estimates will be enough.

Grid, covering and entropy numbers

- (Quasi-)norms on $\mathbb{R}^d$

$$\|x\|_p := \begin{cases} \left(\sum_{j=1}^d |x_j|^p\right)^{1/p} & , \ 0 < p < \infty \\ \max_{1\le j\le d} |x_j| & , \ p = \infty \end{cases}$$

  with (closed) unit balls $B_p^d := \{x \in \mathbb{R}^d : \|x\|_p \le 1\}$

- Let $A \subseteq \mathbb{R}^d$.

  Grid number $\quad G(A) := \#(A \cap \mathbb{Z}^d)$

  Covering numbers $\quad N_\varepsilon(A) :=$ minimal $n \in \mathbb{N}$ such that there are $x_1, \dots, x_n \in \mathbb{R}^d$ with $A \subseteq \bigcup_{i=1}^n (x_i + \varepsilon B_\infty^d)$

  Entropy numbers $\quad \varepsilon_n(A) := \{\inf \varepsilon > 0 : N_\varepsilon(A) \le n\}$

- Here, covering and entropy numbers are always w.r.t. the sup-norm.

Grid numbers vs. covering numbers

- A subset $A \subset \mathbb{R}^d$ is called solid, if
  $$(x_j) \in A \text{ and } |y_j| \le |x_j| \text{ implies } (y_j) \in A.$$
  Examples: rB_p^d for all $r > 0$ and $0 < p \le \infty$

Lemma

Let $A \subseteq \mathbb{R}^d$ be a solid subset and $0 < \varepsilon < 1/2$. Then

$$N_1(A) \le G(A) \le N_\varepsilon(A)\,.$$

- Proof. Given $x \in \mathbb{R}^d$, define $k(x) = (k_j) \in \mathbb{Z}^d$ by $k_j = \text{sign} x_j \cdot [|x_j|]$. Then the set $\{k(x) : x \in A\}$ is a 1-net for A and, since A is solid, it is equal to $A \cap \mathbb{Z}^d$. This proves $N_1(A) \le G(A)$.
  The inequality $G(A) \le N_\varepsilon(A)$ follows from the fact that each ball of radius $\varepsilon < 1/2$ in ℓ_∞^d is a cube of side length $2\varepsilon < 1$, whence it contains at most one element of $\mathbb{Z}^d$. Therefore every covering of A by ε-balls in ℓ_∞^d must have at least $G(A)$ elements.

- Covering numbers are homogeneous, in the sense of

$$N_\varepsilon(A) = N_{\lambda\varepsilon}(\lambda A) \quad \text{for all } \lambda, \varepsilon > 0\,.$$

This is an advantage over grid numbers!

- For large ℓ_p-balls the previous lemma can be improved.

Lemma

Let $0 < p < \infty$, $d \in \mathbb{N}$ *and* $r > d^{1/p}/2$. *Set* $\tilde{p} = \min(1, p)$ *and*

$$\ell = \ell(r,p,d) = (r^{\tilde{p}} - d^{\tilde{p}/p}/2^{\tilde{p}})^{1/\tilde{p}}$$
$$L = L(r,p,d) = (r^{\tilde{p}} + d^{\tilde{p}/p}/2^{\tilde{p}})^{1/\tilde{p}}$$

Then $\quad N_{1/2}(\ell\, B_p^d) \le G(r\, B_p^d) \le N_{1/2}(L\, B_p^d)$

- Prooof: By triangle inequality and volume arguments.
- Note that $\quad \ell(r,p,d) \asymp r \asymp L(r,p,d)$ as $r \to \infty$.

Appoximation in $H^{s,p}(\mathbb{T}^d)$ via entropy of ℓ_p-balls

- Recall that we have already shown

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = (1 + r^p)^{-s/p}$$

for $r \in \mathbb{N}$ and $n = \#\{k \in \mathbb{Z}^d : \|k\|_p \le r\} = G(rB_p^d)$.

- Together with the two lemmata this implies the following result.

Theorem

Let $s > 0$, $0 < p \le \infty$ and $d \in N$. Then, for all $n \in \mathbb{N}$, one has

$$\left(2^{-1-1/p}\varepsilon_n(B_p^d)\right)^s \le a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(4\varepsilon_n(B_p^d)\right)^s$$

- For $p = \infty$ we have $\quad n^{-1/d} \le \varepsilon_n(B_\infty^d) \le 4n^{-1/d}, \quad$ for all $n \in \mathbb{N}$,

  For $0 < p < \infty$ the entropy numbers $\varepsilon_n(B_p^d) = \varepsilon_n(id : \ell_p^d \to \ell_\infty^d)$ are also completely understood.

Entropy numbers of ℓ_p-balls

Lemma (Schütt 1984, Edmunds/Triebel 1996, K. 2001)

Let $0 < p < \infty$ *and* $d \in \mathbb{N}$. *Then*

$$\varepsilon_n(id : \ell_p^d \to \ell_\infty^d) \sim \begin{cases} 1 & , \ 1 \leq n \leq d \\ \left(\frac{\log(1+d/\log n)}{\log n}\right)^{1/p} & , \ d \leq n \leq 2^d \\ d^{-1/p} n^{-1/d} & , \ n \geq 2^d \end{cases}$$

- We have explicit expressions for the constants hidden in $\sim$. In particular, for fixed d, we have

$$\lim_{n\to\infty} n^{1/d} \varepsilon_n(id : \ell_p^d \to \ell_\infty^d) = \frac{vol(B_p^d)^{1/d}}{2}.$$

Asymptotic constants

- Putting everything together, we can show the existence of asymptotically (in n) optimal constants.

Theorem (K./Sickel/Ullrich, J.Complexity 2014)

Let $0 < s, p < \infty$ *and* $d \in \mathbb{N}$. *Then*

$$\lim_{n\to\infty} n^{s/d} a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) = vol(B_p^d)^{s/d} \sim d^{-s/p}$$

- Optimal constant is of order
  $d^{-s/2}$ for the natural norm ($p = 2$)
  $d^{-1/2}$ for the modified natural norm ($p = 2s$)
- We got the correct order $n^{-s/d}$ of the a_n in n and the
  exact decay rate $d^{-s/p}$ of the constants in d.
- Polynomial decay in d of the constants helps in error estimates!

Estimates for large n

Theorem (KSU 2014, case $p = 1$)

Let $s > 0$ and $n \geq 6^d/3$. Then

$$d^{-s} n^{-s/d} \leq a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \leq (4e)^s d^{-s} n^{-s/d} .$$

- We have similar estimates for all other $0 < p < \infty$, but for $p = 1$ the constants are nicer.
- Note the correct d-dependence d^{-s} of the constants!

Estimates for small n

Theorem (KSU 2014)

Let $p = 1$ and $2 \le n \le 2^d$. Then

$$\left(\frac{1}{2+\log_2 n}\right)^s \le a_n(I_d : H^{s,1}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \le \left(\frac{\log_2(2d+1)}{\log_2 n}\right)^s .$$

- This estimate was shown by combinatorial arguments, which only worked for $p = 1$. Using the relation to entropy, we could close the gap between lower and upper bounds and treat arbitrary p's.

Theorem (K/Mayer/Ullrich, preprint 2014)

Let $s > 0$, $0 < p < \infty$ and $2 \le n \le 2^d$. Then

$$a_n(I_d : H^{s,p}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)) \sim \left(\frac{\log_2(1 + d/\log_2 n))}{\log_2 n}\right)^{s/p} .$$

(We have explicit expressions for the hidden constants.)

Thank you for your attention!