# Tractability Using Periodized Generalized Faure Sequences

Christiane Lemieux

Department of Statistics and Actuarial Science
University of Waterloo, Canada

Information-Based Complexity and Stochastic Computation Workshop

ICERM, September 15, 2014

# Motivation

- Faure sequences and their generalizations are often used because they achieve the optimal value of 0 for the quality parameter $t$.
- Problem is they are not extensible in the dimension, since the base $b$ must be at least as large as the dimension $s$ to achieve $t = 0$
- Can we make them extensible in the dimension, and if so, for what kind of problems would that work?

Based on

** C. Lemieux, H. Faure, New perspectives on $(0, s)$-sequences, in: P. L'Ecuyer, A. Owen (Eds.), Monte Carlo and Quasi-Monte Carlo Methods 2008, Springer-Verlag, 2009, pp. 113–130.*

** C. Lemieux, Tractability using periodized generalized Faure sequences, to appear in Journal of Complexity, 2014.*

# Outline

1. Problem Setup and Review
2. Periodized Generalized Faure Sequences
3. Tractability Results
4. Numerical Results

# 1 – Problem Setup and Review

- Goal is to estimate
$$I_s(f) = \int_{[0,1)^s} f(\mathbf{x}) d\mathbf{x}$$
where $f : I^s = [0,1)^s \to \mathbb{R}$ is a real-valued function.
- Monte Carlo or (randomized) quasi-Monte Carlo estimate $I_s(f)$ by
$$Q_{N,s}(f, P_N) = \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i)$$
where $P_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subseteq [0,1)^s$.
- With (r)QMC, $P_N$ is a low-discrepancy point set:
$$E(P_N, \mathbf{z}) = \frac{1}{N} \sum_{i=1}^{N} \prod_{j=1}^{s} \mathbf{1}_{x_{i,j} \le z_j} - \prod_{j=1}^{s} z_j,$$
then low (star-)discrepancy means
$$D^*(P_N) = \sup_{\mathbf{z} \in I^s} |E(P_N, \mathbf{z})| \in O(N^{-1} \log^s N).$$

- Example: want to estimate the exp. nb of clients waiting more than 5 minutes in a bank over fixed horizon $T$; fcts $g$ and $h$ resp. generate inter-arrival and service times $A_i, S_i$, so

$$f(\mathbf{x}) = \phi(\mathbf{Y}) = \text{nb of clients who waited more than 5 minutes}$$
$$\mathbf{Y} = (A_1 = g(x_1), S_1 = h(x_2), \ldots, A_{\mathcal{N}} = g(x_{2\mathcal{N}-1}), S_{\mathcal{N}} = h(x_{2\mathcal{N}}))$$
$$\mathcal{N} = \text{number of clients observed over horizon } [0, T]$$

  Here $s = 2\mathcal{N}$ is not known ahead of time.
- In such cases, we need $P_N$ to be extensible in the dimension, i.e., coordinates for each $\mathbf{x}_i$ can be added "on the fly".

# Review of Faure sequences

- *Elementary intervals in base b:* subsets of $I^s$ of the form

$$\prod_{j=1}^{s}\left[\frac{l_j}{b^{r_j}},\frac{l_j+1}{b^{r_j}}\right) \qquad \text{(volume is } b^{-M} \text{ where } M = r_1 + \ldots + r_s)$$

where $0 \le l_j < b^{r_j}$ and $r_j \ge 0$, $j = 1, \ldots, s$.

$M = 5$

- A $(0, m, s)$-net in base $b$ is a point set $P_N$ with $N = b^m$ points s.t. any elementary interval of volume $b^{-M}$ contains $b^{m-M}$ pts from $P_N$, when $M \le m$.
- A $(0, s)$-sequence in base $b$ is a sequence of points $\mathbf{x}_1, \mathbf{x}_2, \ldots$ such that $\{\mathbf{x}_{kb^m+1}, \ldots, \mathbf{x}_{(k+1)b^m}\}$ is a $(0, m, s)$-net for all $m \ge 0$ and all $k \ge 0$.

- *Linearly scrambled van der Corput sequence in base b:* For a prime base $b$, it is obtained by choosing a matrix $C = (c_{r,k})_{r,k\geq 1}$ with elements in $\mathbb{Z}_b$ and an infinite number of rows and columns, and then defining the $n$th term of this sequence as

$$S_b^C(n) := \sum_{r=0}^{\infty} \frac{y_{n,r}}{b^{r+1}} \quad \text{in which} \quad y_{n,r} = \sum_{k=0}^{\infty} c_{r+1,k+1} \cdot a_k(n),$$

  where digits $a_{k(n)}$ come from $n = \sum_{k\geq 0} a_k(n) b^k$.
- Sequence of points in $I^s$ can be constructed using $(S_b^{C_1}, \ldots, S_b^{C_s})$, where $C_1, \ldots, C_s$ are *generating matrices*.
- Original Faure sequences (1982): $C_j = P_b^{j-1}$, where $P_b$ is the (upper triangular) Pascal matrix $P_b$ in $\mathbb{Z}_b \Rightarrow (0, s)$-sequence if $b \geq s$.
- Generalized by Tezuka (1994) to $C_j = A_j P_b^{j-1}$, where each $A_j$ is an (NLT) matrix: also a $(0, s)$-sequence if $b \geq s$.

# Insight from Korobov Lattices

- Korobov lattices are extensible in the dimension, but coordinates start repeating if $s \geq N$:

$$P_N = \left\{ \frac{i-1}{N}(1, a, a^2 \bmod N, \ldots, a^{s-1} \bmod N) \bmod 1, i = 1, \ldots, N \right\}.$$

- Can take $N$ prime and $a$ primitive element modulo $N$ to get cycle of maximal period $N-1$.
- With $s \geq N$ means projections over indices with lag of $N-1$ are bad, but $N$ large can mitigate this issue since corresponding projections of $f$ are often not important in that case.
- Bank example where $s$ is not fixed but on average equal to 2000; use Korobov lattice with $N = 1021$ and $a = 76$

| | MC | Kor | Sobol' |
|---|---|---|---|
| $Q_{N,s}(f, P_N)$ | 543 | 544 | 544 |
| HW | 2.21 | 0.93 | 0.81 |

# 2 – Periodized Generalized Faure Sequences (PGFS)

- Fix the base $b$, and for any $s \geq 1$, use sequence defined over $\mathbb{Z}_b$ based on generating matrices

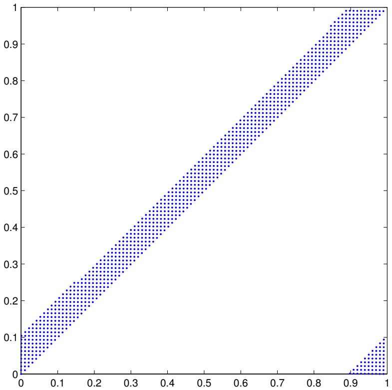
$$C_j = A_j P_b^{j-1}, \qquad j = 1, \ldots, s \text{ (can be } \geq b).$$

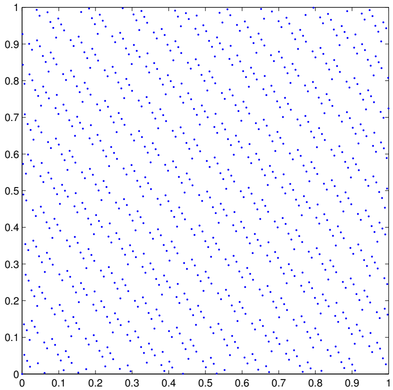
- Scrambling matrices $A_j$ fixed as follows: choose a period $p \approx b/2$ and then let

$$A_j = f_{j \bmod p} I \qquad \text{(diagonal)}$$

where the multipliers $f_j \in \mathbb{Z}_b$ are ordered according to the quality of the one-dimensional van der Corput sequence they generate (similar idea used to define generalized Faure sequences in LF09).

# (49th,50th) coordinates of 1000 first points of Faure (left) and PGFS with $b = 97$



Of course, if $s \geq b$, $t > 0$ for a PGFS, but we have $t_b = 0$, where:

**Definition**

For sequence $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ over $\mathbb{Z}_b$, $t_k{=}0$ if for each $u = \{j_1, \ldots, j_r\}$ satisfying $1 \leq j_1 < \cdots < j_r \leq s$, $1 \leq |u| = r \leq k$, and $r(u) := j_r - j_1 + 1 \leq k$, the corresponding projection

$$\{(x_{n,j_1}, x_{n,j_2}, \ldots, x_{n,j_r}), n \geq 1\}$$

is a $(0, r)$-sequence over $\mathbb{Z}_b$.

**Result:** *$P_N$ the first $N$ points of a digital sequence over $\mathbb{Z}_b$ with $t_b = 0$; $P_N^u$ the projection of $P_N$ over $u \subseteq \{1, \ldots, s\}$ with $r(u) \leq b$, then*

$$D^*(P_N^u) \leq \frac{1}{N} \frac{b+1}{2b} (b \log_b(bN))^{|u|}.$$

# 3 – Tractability Results

- Look at worst-case error over Hilbert space $H_s$ with norm $\| \cdot \|_{H_s}$:

$$e(P_N; H_s) = \sup\{|I_s(f) - Q_{N,s}(f; P_N)| : f \in H_s, \|f\|_{H_s} \leq 1\},$$

  and compare it with the initial error, defined as

$$e(0; H_s) = \sup\{|I_s(f)| : f \in H_s, \|f\|_{H_s} \leq 1\}.$$

  We then define $n(\epsilon, H_s)$ as the smallest $n$ for which there exists $P_n$ such that $e(P_n; H_s) \leq \epsilon \cdot e(0; H_s)$, where $\epsilon$ is in $(0, 1)$.

- Integration over $H_s$ is said to be *QMC-tractable* if there exist non-negative numbers $C$, $p$, and $q$ such that

$$n(\epsilon, H_s) \leq C\epsilon^{-p} s^q \qquad \text{for all } \epsilon \in (0,1) \text{ and all } s \geq 1. \qquad (1)$$

- If $q = 0$ in (1), then integration over $H_s$ is *QMC-strongly tractable*, and the infimum of the numbers $p$ satisfying (1) with $q = 0$ is called the $\epsilon-$*exponent of QMC-strong tractability*.

- Can adapt the results and proofs from SWW04 to our settings.
- As in SWW04, we assume $H_s$ is a reproducing kernel Hilbert space with a reproducing kernel of the form

$$K_s(\mathbf{x}, \mathbf{y}) = \sum_{u \subseteq \{1,\ldots,s\}} \gamma_{s,u} \prod_{j \in u} \eta_j(x_j, y_j), \tag{2}$$

  and the weights $\gamma_{s,u}$ are arbitrary non-negative numbers. Hence any $f \in H_s$ satisfies $f(\cdot) = \langle f(\mathbf{x}), K_s(\mathbf{x}, \cdot)\rangle$.
- Case A: *anchored Sobolev space $H(K_{s,A})$*: take $\eta_{j,A}(x, y) = \min(|x_j - a_j|, |y - a_j|)$ if $(x - a_j)(y - a_j) > 0$ and 0 otherwise. The point $(a_1, \ldots, a_s)$ is called the *anchor*.
- Case B: *unanchored Sobolev space $H(K_{s,B})$*: take $\eta_{j,B}(x, y) = \frac{1}{2} B_2(|x - y|) + \left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right)$, where $B_2(\cdot)$ is the Bernoulli polynomial of degree 2, i.e., $B_2(x) = x^2 - x + \frac{1}{6}$

# Choice of weights

- Tractability over weighted spaces usually occurs by choosing weights of the form $\gamma_{s,u} = \prod_{j \in u} \gamma_j$ for some $0 \leq \gamma_j \leq 1, j = 1, \ldots, s$, or when there exists an integer $r$ such that

$$\gamma_{s,u} = 0 \text{ for all } u \text{ with } |u| > r \text{ (finite order)}.$$

- Here, we propose a special case of the latter, which makes use of the notion of range $r(u)$ defined earlier.

**Definition**

A set of weights $\{\gamma_{s,u}\}_{u \subseteq \{1,\ldots,s\}}$ is said to be of *finite-range* if there exists an integer $R \in \{0, \ldots, s-1\}$ (called the range) such that

$$\gamma_{s,u} = 0 \text{ if } r(u) = \max_j\{j \in u\} - \min_j\{j \in u\} + 1 > R.$$

**Theorem:** *Let $H(K_s)$ be the anchored Sobolev space $H(K_{s,A})$ with an arbitrary anchor* $\mathbf{a}$*, or the unanchored Sobolev space $H(K_{s,B})$, and assume we have finite-range weights $\{\gamma_{s,u}\}_{u \subseteq \{1,\ldots,s\}}$ of range $R \geq 1$. Let $P_N$ be the first $N$ points of a digital sequence over $\mathbb{Z}_b$ such that $t_R = 0$, where $R \leq b$. Then*

$$e^2(P_N; H(K_s)) \leq \frac{1}{N^2} \sum_{\substack{\emptyset \neq u \subseteq \{1,\ldots,s\} \\ r(u) \leq R}} \gamma_{s,u} \left( \frac{b+1}{2b} \right)^2 (2b \log_b bN)^{2|u|}. \qquad (3)$$

**Theorem:** *Let* $\{\gamma_{s,u}\}_{s\geq 1, u\subseteq\{1,\ldots,s\}}$ *be weights of finite-range* $R \geq 1$. *Let* $P_N$ *be the first* $N$ *points of a digital sequence over* $\mathbb{Z}_b$ *with* $b \geq R$ *and such that* $t_R = 0$ *for all* $s \geq 1$. *(a) Consider the anchored Sobolev space* $H(K_{s,A})$ *with arbitrary anchor* $\mathbf{a}$ *and weights* $\gamma_{s,u}$. *Then*

$$\frac{e(P_N; H(K_{s,A}))}{e(0; H(K_s))} \leq C(b)\frac{1}{N}(\log_b bN)^b,$$

*where* $C(b) = (4\sqrt{3}b)^b$. *Furthermore, for any arbitrary* $\delta > 0$ *there exists a constant* $C_\delta$ *independent of* $s$ *and* $N$ *such that*

$$\frac{e(P_N; H(K_{s,A}))}{e(0; H(K_{s,A}))} \leq C_\delta N^{-1+\delta}.$$

*Hence we have QMC-strong tractability with* $\epsilon-$*exponent 1.*

*(b) Unanchored Sobolev space $H(K_{s,B})$ with weights $\gamma_{s,u}$;*
*(i) If there exists $c_B^*$ such that $\gamma_{s,u} \leq c_B^*$ for all $u$ and $s \geq 1$, then*

$$\frac{e(P_N; H(K_{s,B}))}{e(0; H(K_{s,B}))} \leq C_1(b) \frac{s^{1/2}}{N} (\log_b bN)^b,$$

*where $C_1(b) = \sqrt{c_B^*}(\sqrt{8}b)^b$ (so QMC-tractable with $q = 1/2$).*
*(ii) If we further assume that $\mathcal{M} = \sup_{s=1,2,\ldots} \sum_{u:0 \leq r(u) \leq R} \gamma_{s,u} < \infty$ then*

$$\frac{e(P_N; H(K_{s,B}))}{e(0; H(K_{s,B}))} \leq C_2(b) \frac{1}{N} (\log_b bN)^b,$$

*where $C_2(b) = \sqrt{\mathcal{M}}(2b)^b$. Furthermore, for any arbitrary $\delta > 0$ there exists a constant $C_{B,\delta}$ independent of $s$ and $N$ such that*

$$\frac{e(P_N; H(K_{s,B}))}{e(0; H(K_{s,B}))} \leq C_{B,\delta} N^{-1+\delta}.$$

*Hence we have QMC-strong tractability with $\epsilon-$exponent 1.*

# 4 – Numerical results

- Want to demonstrate that PGFS can compete with good/popular choices
- Will compare with Sobol', extensible rank-1 lattice of CKN06, extensible Korobov lattice (HHLL01 and GL07)
- Will use 3 different problems: test-function with finite-range weights, financial option, queueing example with unbounded $s$
- Use RQMC (shift for lattices, digital shift for Sobol' and PGFS); can then look, e.g., at

$$\frac{1}{m}\sum_{l=1}^{m}|Q_{N,s}(f,\tilde{P}_N(l)) - I_s(f)| \text{ (aver. absol. err.)}$$

or

$$\frac{1}{m}\sum_{l=1}^{m}(Q_{N,s}(f,\tilde{P}_N(l)) - \bar{Q}_{N,s})^2 \text{ (estim. var.)}$$

# (i) – Test-function with finite-range weights

Based on the test function from Sobol' and Asotsky

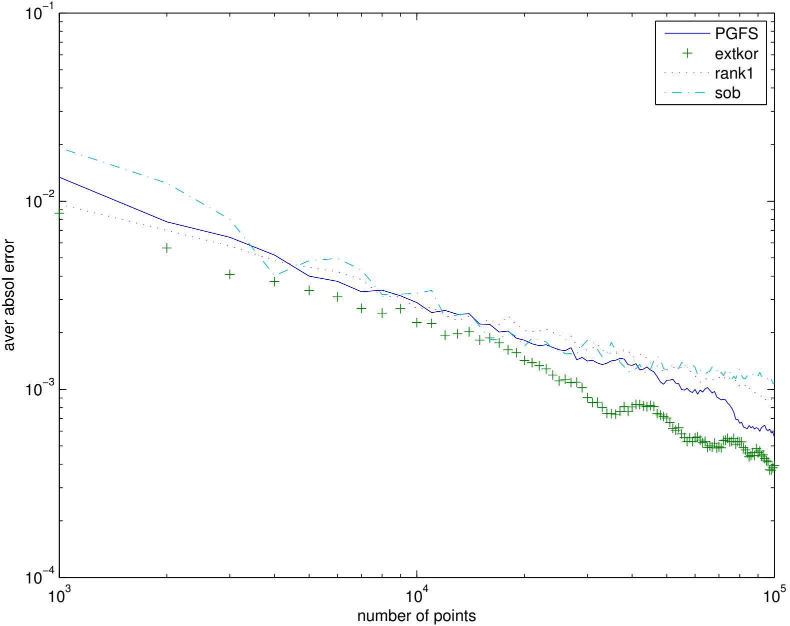
$$g(\mathbf{x}) = \prod_{j=1}^{s} (1 + c(x_j - 0.5)).$$

Here however, we slightly modify this function and use instead

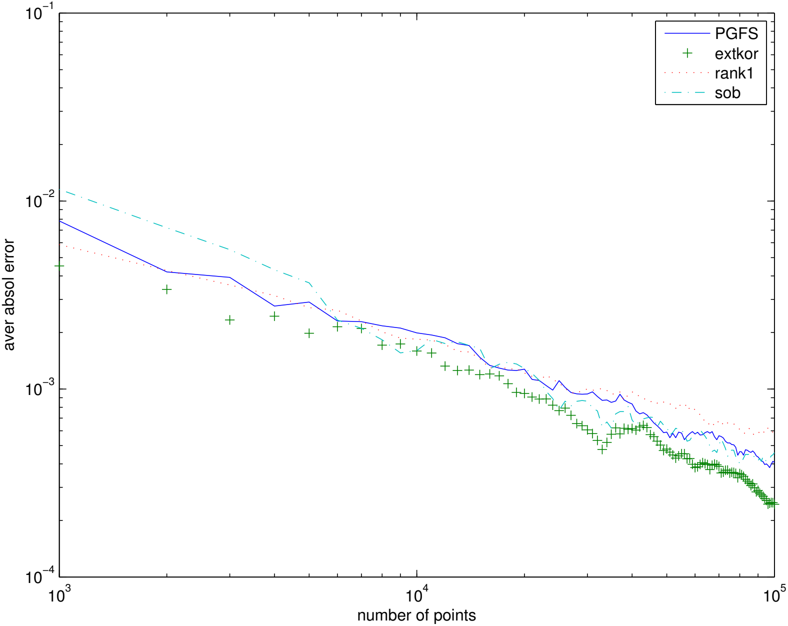
$$g_{k,s}(\mathbf{x}) = \frac{1}{s-k+1} \sum_{l=1}^{s-k+1} \underbrace{g(x_l, \ldots, x_{l+k-1})}_{range \;\; k}$$

for different values of $s$ and $k$, so weights are of range $k$.

# Example with $g_{20,96}$ – PGFS with $b = 97, p = 42$



# Example with $g_{20,250}$ – PGFS with $b = 97$, $p = 42$



# (ii) – Asian call option

- Corresponding function $f(\mathbf{x})$ is such that option price $C(T, s, r, \sigma)$ satisfies

$$C(T, s, r, \sigma) = \mathrm{E}\left(e^{-rT} \max\left(\frac{1}{s}\sum_{j=1}^{s} S(t_j) - K, 0\right)\right) = \int_{I^s} f(\mathbf{x}) d\mathbf{x},$$

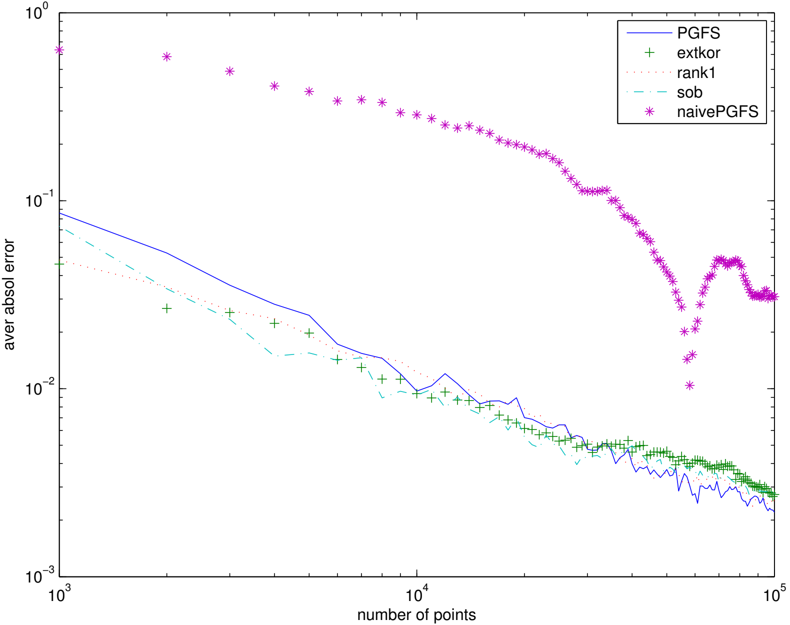
where $S(t_j)$ is the price of the underlying asset at time $t_j = jT/s$.

- Assume $S(t_j)|S(t_{j-1}) \sim \mathcal{LN}((r - \sigma^2/2)/s, \sigma^2/s)$, where $T$ is the expiration time of the option, $r$ is the risk-free rate, and $\sigma$ is the volatility of the asset.
- Hence the function $f(\mathbf{x})$ in this case can be written as

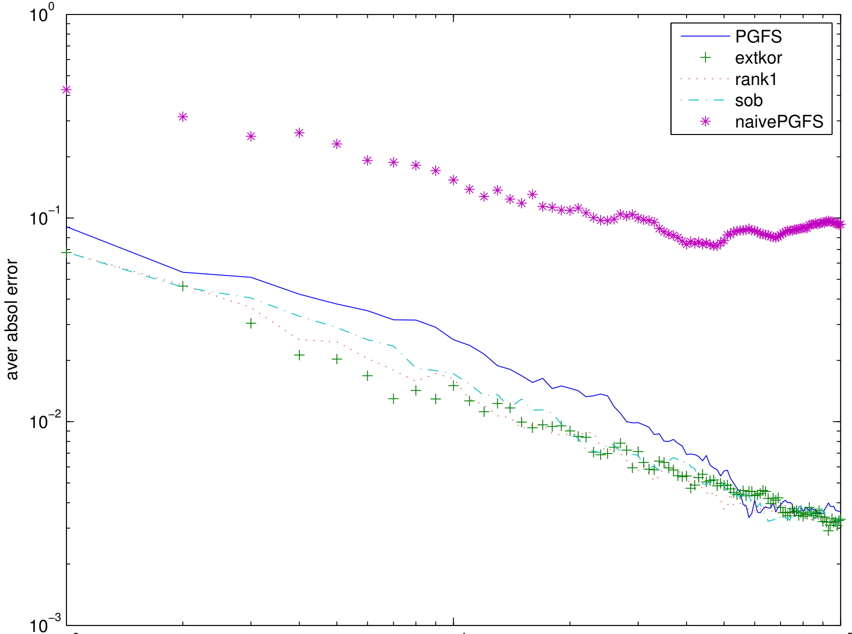
$$f(\mathbf{x}) = e^{-rT}\max\left(0, \frac{1}{s}\sum_{j=1}^{s} \exp((r - \sigma^2/2)(jT/s) + \sum_{l=1}^{j} \Phi^{-1}(x_l)\sigma\sqrt{\frac{T}{s}}\right)$$

where $\Phi(\cdot)$ is the CDF of a standard Normal rv.

# Asian option with $s = 64$ – PGFS with $b = 241$ and $p = 122$

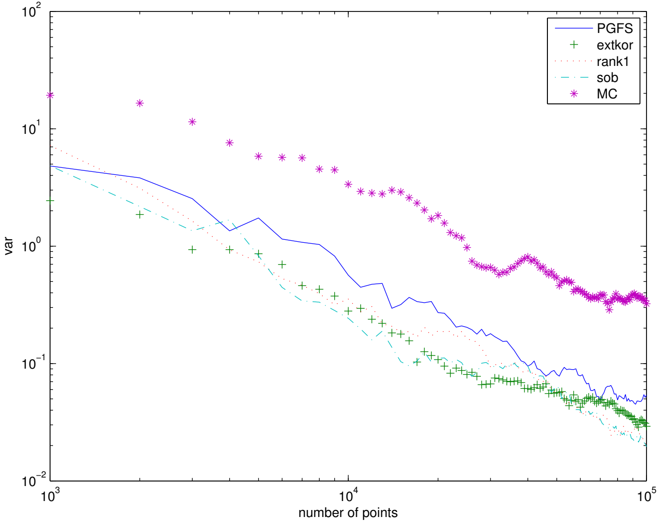


# Asian option with $s = 256$



# (iii)- Bank example with unbounded $s$

Same as motivating example (i.e., average $s$ is 2000): added MC for comparison, PGFS with $b = 727$ and $p = 396$



# Some References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*Support from National Science and Engineering Research Council (NSERC) of Canada is acknowledged.*