Plan

0. Prologue: bifurcation currents for rational maps.
1. Stability/bifurcation dichotomy for Kleinian groups.
2. Lyapunov exponent of a surface group representation
3. The degree of a projective structure.

Joint work (in progress) with Bertrand Deroin (Orsay)
Prologue: bifurcation currents for rational maps.

Let \((f_\lambda)_{\lambda \in \Lambda}\) be a holomorphic family of rational mappings of degree \(d\) on \(\mathbb{P}^1\).
Let $(f_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of rational mappings of degree $d$ on $\mathbb{P}^1$. For every $\lambda$, $f_\lambda$ admits a natural invariant probability measure $\mu_\lambda$ (Brolin-Lyubich measure). We can consider the Lyapunov exponent function.

$$ \lambda \mapsto \chi(f_\lambda) = \int \log \| (f_\lambda)'(z) \| \, d\mu_\lambda(z) $$
Let \((f_{\lambda})_{\lambda \in \Lambda}\) be a holomorphic family of rational mappings of degree \(d\) on \(\mathbb{P}^1\).

For every \(\lambda\), \(f_{\lambda}\) admits a natural invariant probability measure \(\mu_{\lambda}\) (Brolin-Lyubich measure).

We can consider the Lyapunov exponent function.

\[
\lambda \mapsto \chi(f_{\lambda}) = \int \log \| (f_{\lambda})'(z) \| \, d\mu_{\lambda}(z)
\]

It is continuous and plurisubharmonic* \((psh)\) on \(\Lambda\).
Let \((f_\lambda)_{\lambda \in \Lambda}\) be a holomorphic family of rational mappings of degree \(d\) on \(\mathbb{P}^1\).

For every \(\lambda\), \(f_\lambda\) admits a natural invariant probability measure \(\mu_\lambda\) (Brolin-Lyubich measure).

We can consider the **Lyapunov exponent function**.

\[
\lambda \mapsto \chi(f_\lambda) = \int \log \|(f_\lambda)'(z)\| \; d\mu_\lambda(z)
\]

It is continuous and plurisubharmonic\(^*\) (psh) on \(\Lambda\).

**Definition (DeMarco)**

The bifurcation current is \(T_{\text{bif}} = dd^c_\lambda(\chi(f_\lambda))\)
Prologue: bifurcation currents for rational maps.

Results.
Prologue: bifurcation currents for rational maps.

Results.

- **Support theorem (DeMarco)** $\text{Supp}(T_{\text{bif}})$ is the bifurcation locus of the family.
Results.

- **Support theorem (DeMarco)** $\text{Supp}(T_{\text{bif}})$ is the bifurcation locus of the family.

- **Equidistribution of special subvarieties. (D.-Favre, Bassanelli-Berteloot)** Natural sequences of hypersurfaces associated with bifurcations equidistribute towards $T_{\text{bif}}$. 
Prologue: bifurcation currents for rational maps.

Results.

- Support theorem (DeMarco) $\text{Supp}(\mathcal{T}_{\text{bif}})$ is the bifurcation locus of the family.
- Equidistribution of special subvarieties. (D.-Favre, Bassanelli-Berteloot) Natural sequences of hypersurfaces associated with bifurcations equidistribute towards $\mathcal{T}_{\text{bif}}$.
- Formulas for Lyapunov exponent.
  Example: the Manning-Przytycki formula: if $f_\lambda$ is a monic polynomial of degree $d$ then
  \[
  \chi(f_\lambda) = \log d + \sum_{c \text{ critical}} G_{f_\lambda}(c).
  \]
Aim : translate these concepts into the context of families of subgroups of $\text{Aut}(\mathbb{P}^1) = \text{PSL}(2, \mathbb{C})$. 
1. Stability/bifurcation dichotomy for Kleinian groups

- \( \text{Aut}(\mathbb{P}^1) \cong \text{PSL}(2, \mathbb{C}) \) via \( \frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Fix \( \| \cdot \| \) on \( \text{SL}(2, \mathbb{C}) \).
1. Stability/bifurcation dichotomy for Kleinian groups

- $\text{Aut}(\mathbb{P}^1) \simeq \text{PSL}(2, \mathbb{C}) \text{ via } \frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Fix $\| \cdot \|$ on $\text{SL}(2, \mathbb{C})$.

- A Möbius transformation $\gamma(z) = \frac{az+b}{cz+d} \neq \text{id}$ has a type
  - elliptic model: $z \mapsto e^{i\theta}z$, $\text{tr}^2(\gamma) \in [0, 4)$
  - parabolic model: $z \mapsto z + 1$, $\text{tr}^2(\gamma) = 4$
  - loxodromic model: $z \mapsto kz$, $\text{tr}^2(\gamma) \notin [0, 4]$
1. Stability/bifurcation dichotomy for Kleinian groups

Let $G$ be a finitely generated group, $\Lambda$ a complex manifold, and $
abla = (\rho_{\lambda})_{\lambda \in \Lambda} : \Lambda \times G \to \text{PSL}(2, \mathbb{C})$ be a holomorphic family of representations of $G$ (i.e. it is holomorphic in $\lambda$ and a homomorphism in $g$).
1. Stability/bifurcation dichotomy for Kleinian groups

Let $G$ be a finitely generated group, $\Lambda$ a complex manifold, and $\rho = (\rho_\lambda)_{\lambda \in \Lambda} : \Lambda \times G \rightarrow \text{PSL}(2, \mathbb{C})$ be a holomorphic family of representations of $G$ (i.e. it is holomorphic in $\lambda$ and a homomorphism in $g$).

Standing assumptions:

(R1) the family is non-trivial
(R2) $\rho_{\lambda_0}$ is faithful for some $\lambda_0$
(R3) for every $\lambda$, $\rho_\lambda$ is non-elementary (i.e. does not have a finite orbit on $\mathbb{H}^3 \cup \mathbb{P}^1$).
1. Stability/bifurcation dichotomy for Kleinian groups

Let $G$ be a finitely generated group, $\Lambda$ a complex manifold, and $\rho = (\rho_\lambda)_{\lambda \in \Lambda} : \Lambda \times G \to \text{PSL}(2, \mathbb{C})$ be a holomorphic family of representations of $G$ (i.e. it is holomorphic in $\lambda$ and a homomorphism in $g$).

Standing assumptions:

(R1) the family is non-trivial
(R2) $\rho_{\lambda_0}$ is faithful for some $\lambda_0$
(R3) for every $\lambda$, $\rho_\lambda$ is non-elementary (i.e. does not have a finite orbit on $\mathbb{H}^3 \cup \mathbb{P}^1$).

(or sometimes
(R3') there is $\lambda_0$, s.t. $\rho_{\lambda_0}$ is non-elementary.)
1. Stability/bifurcation dichotomy for Kleinian groups

**Theorem (Sullivan, and also Bers, Marden, etc.)**

Let \((\rho_\lambda)_{\lambda \in \Lambda}\) be as above, and \(\Omega \subset \Lambda\) be a connected open subset. The following are equivalent:

1. \(\forall \lambda \in \Omega, \rho_\lambda\) is discrete;
2. \(\forall \lambda \in \Omega, \rho_\lambda\) is faithful;
3. for every \(g \in G\), \(\rho_\lambda(g)\) does not change type as \(\lambda\) ranges in \(\Omega\);
4. for all \(\lambda, \lambda' \in \Omega\), \(\rho_\lambda\) and \(\rho_{\lambda'}\) are quasiconformally conjugate on \(\mathbb{P}^1\), i.e. there exists a qc homeo \(\phi : \mathbb{P}^1 \to \mathbb{P}^1\) s.t. \(\forall g \in G\), \(\rho_{\lambda_0}(g) \circ \phi = \phi \circ \rho_{\lambda_1}(g)\).
Theorem (Sullivan, and also Bers, Marden, etc.)

Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be as above, and $\Omega \subset \Lambda$ be a connected open subset. The following are equivalent:

1. $\forall \lambda \in \Omega$, $\rho_\lambda$ is discrete;
2. $\forall \lambda \in \Omega$, $\rho_\lambda$ is faithful;
3. for every $g \in G$, $\rho_\lambda(g)$ does not change type as $\lambda$ ranges in $\Omega$;
4. for all $\lambda, \lambda' \in \Omega$, $\rho_\lambda$ and $\rho_{\lambda'}$ are quasiconformally conjugate on $\mathbb{P}^1$, i.e. there exists a qc homeo $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ s.t. $\forall g \in G$, $\rho_{\lambda_0}(g) \circ \phi = \phi \circ \rho_{\lambda_1}(g)$.

Such a family is said to be stable on $\Omega$. 
1. Stability/bifurcation dichotomy for Kleinian groups

So we get a decomposition

$$\Lambda = \text{Stab} \cup \text{Bif}$$

as a maximal domain of local stability and its complement.
1. Stability/bifurcation dichotomy for Kleinian groups

So we get a decomposition

$$\Lambda = \text{Stab} \cup \text{Bif}$$

as a maximal domain of local stability and its complement.

We also have identified a dense codimension 1 phenomenon responsible for bifurcations:

**Corollary**

For every $t \in [0, 4]$,

$$\bigcup_{g \in G} \{ \lambda, \; \text{tr}^2(\rho_\lambda(g)) = t \} \supset \text{Bif}. $$
1. Stability/bifurcation dichotomy for Kleinian groups

So we get a decomposition

\[ \Lambda = \text{Stab} \cup \text{Bif} \]

as a maximal domain of local stability and its complement.

We also have identified a dense codimension 1 phenomenon responsible for bifurcations:

**Corollary**

For every \( t \in [0, 4], \)

\[ \bigcup_{g \in G} \{ \lambda, \ tr^2(\rho_\lambda(g)) = t \} \supset \text{Bif}. \]

**Note:** Bif has non-empty interior (Margulis-Zassenhaus lemma) so Stab is not dense in this setting.
For the remainder of the talk $G = \pi_1(X, \star)$ is the fundamental group of a compact connected surface of genus $\geq 2$, **endowed with Riemann surface structure.**

Endow $X$ with its Poincaré metric.
For the remainder of the talk $G = \pi_1(X, \star)$ is the fundamental group of a compact connected surface of genus $\geq 2$, endowed with Riemann surface structure.

Endow $X$ with its Poincaré metric.

Note: everything should work for (hyperbolic) surfaces of finite type (technically much more difficult)
Let $\rho : G \to \text{PSL}(2, \mathbb{C})$ be a non-elementary representation.

For $v$ a unit tangent vector at $\star$, let $\gamma_{\star, v}$ the corresponding unit speed half geodesic. For $t > 0$ close the path $\gamma_{\star, v}|_{[0, t]}$ by a path of length $\leq \text{diam}(X)$ returning to $\star$. We get a loop $\tilde{\gamma}_t$. 
2. Lyapunov exponent of a surface group representation

Let $\rho : G \to \text{PSL}(2, \mathbb{C})$ be a non-elementary representation. For $v$ a unit tangent vector at $\star$, let $\gamma_{\star,v}$ the corresponding unit speed half geodesic. For $t > 0$ close the path $\gamma_{\star,v}|_{[0,t]}$ by a path of length $\leq \text{diam}(X)$ returning to $\star$. We get a loop $\tilde{\gamma}_t$.

**Definition-Proposition**

For a.e. $v \in T^1_\star X$, the limit

$$\chi_{\text{geodesic}}(\rho) = \lim_{t \to \infty} \frac{1}{t} \log \| \rho ([\tilde{\gamma}_t]) \|$$

exists and does not depend on $v$. 

Romain Dujardin
Bers, Brown and Lyapunov
Let \( \rho : G \to \text{PSL}(2, \mathbb{C}) \) be a non-elementary representation. For \( \nu \) a unit tangent vector at \( \star \), let \( \gamma_{\star, \nu} \) the corresponding unit speed half geodesic. For \( t > 0 \) close the path \( \gamma_{\star, \nu}|_{[0,t]} \) by a path of length \( \leq \text{diam}(X) \) returning to \( \star \). We get a loop \( \tilde{\gamma}_t \).

**Definition-Proposition**

For a.e. \( \nu \in T^1_\star X \), the limit

\[
\chi_{\text{geodesic}}(\rho) = \lim_{t \to \infty} \frac{1}{t} \log \| \rho ([\tilde{\gamma}_t]) \|
\]

exists and does not depend on \( \nu \).

“"The Lyapunov exponent of \( \rho \) associated to geodesic flow on \( X""
Example (Fuchsian representation)

View $X$ as $\Gamma \setminus \mathbb{H}^2$, where $\Gamma$ is a Fuchsian group. Identify $G$ and $\Gamma$ via $\rho_{\text{Fuchs}}$. Then $\chi_{\text{geodesic}}(\rho_{\text{Fuchs}}) = \frac{1}{2}$. 
Example (Fuchsian representation)

View $X$ as $\Gamma \backslash \mathbb{H}^2$, where $\Gamma$ is a Fuchsian group. Identify $G$ and $\Gamma$ via $\rho_{\text{Fuchs}}$. Then $\chi_{\text{geodesic}}(\rho_{\text{Fuchs}}) = \frac{1}{2}$.

Indeed, let $0$ be the lift of $\star$. In $\mathbb{H}^2$, travel a geodesic issued from $0$ for time $t$, and close it by a short path joining its endpoint to $\gamma(0)$ for some $\gamma \in \Gamma$. Then

$$\log \|\gamma\| \simeq \frac{1}{2} \log d_{\mathbb{H}^2}(0, \gamma(0)) = \frac{t}{2} + O(1).$$
Let \((\rho_\lambda)_{\lambda \in \Lambda}\) be a holomorphic family of representations of \(G = \pi_1(X)\) into \(\text{PSL}(2, \mathbb{C})\), satisfying \((R1 - 3)\).

**Theorem**

\[ \lambda \mapsto \chi_{\text{geodesic}}(\lambda) \text{ is a continuous positive psh function on } \Lambda. \]
2. Lyapunov exponent of a surface group representation

Let \((\rho_\lambda)_{\lambda \in \Lambda}\) be a holomorphic family of representations of 
\(G = \pi_1(X)\) into \(\text{PSL}(2, \mathbb{C})\), satisfying \((R1 - 3)\).

**Theorem**

\(\lambda \mapsto \chi_{\text{geodesic}}(\lambda)\) is a continuous positive psh function on \(\Lambda\).

Let \(T_{\text{bif}} = \frac{1}{2}dd^c(\chi_{\text{geodesic}})\) be the bifurcation current.
Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of $G = \pi_1(X)$ into $\text{PSL}(2, \mathbb{C})$, satisfying $(R1 - 3)$.

**Theorem**

$\lambda \mapsto \chi_{\text{geodesic}}(\lambda)$ is a continuous positive psh function on $\Lambda$.

Let $T_{\text{bif}} = \frac{1}{2}dd^c(\chi_{\text{geodesic}})$ be the bifurcation current.

**Theorem**

$\text{Supp}(T_{\text{bif}})$ is the bifurcation locus of the family.
2. Lyapunov exponent of a surface group representation

We also have “equidistribution of special subvarieties”.

Romain Dujardin
Bers, Brown and Lyapunov
We also have “equidistribution of special subvarieties”.

Let $\gamma$ be a closed geodesic on $X$. It defines a conjugacy class in $G$ so $\text{tr}^2(\rho([\gamma]))$ is well defined. For $t \in \mathbb{C}$, let

$$Z(\gamma, t) = \{ \lambda \in \Lambda, \ \text{tr}^2(\rho_\lambda([\gamma]) = t \}.$$ 

(interesting values : $t = 4$, $t = 4 \cos^2(\theta)$, $\theta \notin \pi\mathbb{Q}$.)
2. Lyapunov exponent of a surface group representation

We also have “equidistribution of special subvarieties”. Let $\gamma$ be a closed geodesic on $X$. It defines a conjugacy class in $G$ so $\text{tr}^2(\rho([\gamma]))$ is well defined. For $t \in \mathbb{C}$, let

$$Z(\gamma, t) = \{ \lambda \in \Lambda, \; \text{tr}^2(\rho_\lambda([\gamma])) = t \}.$$ 

(interesting values : $t = 4$, $t = 4 \cos^2(\theta)$, $\theta \notin \pi \mathbb{Q}$.)

Pre-Theorem
For every $t \in \mathbb{C}$, if $(\gamma_n)$ is a random sequence of geodesics with length $\to \infty$, then

$$\frac{1}{4 \text{length}(\gamma_n)}[Z(\gamma_n, t)] \xrightarrow{n \to \infty} T_{\text{bif}}.$$
2. Lyapunov exponent of a surface group representation

We also have “equidistribution of special subvarieties”.
Let \( \gamma \) be a closed geodesic on \( X \). It defines a conjugacy class in \( G \) so \( \text{tr}^2(\rho([\gamma])) \) is well defined. For \( t \in \mathbb{C} \), let

\[
Z(\gamma, t) = \left\{ \lambda \in \Lambda, \; \text{tr}^2(\rho_\lambda([\gamma])) = t \right\}.
\]

(interesting values : \( t = 4, \; t = 4 \cos^2(\theta), \; \theta \notin \pi \mathbb{Q} \).)

Pre-Theorem
For every \( t \in \mathbb{C} \), if \( (\gamma_n) \) is a random sequence of geodesics with length \( \to \infty \), then

\[
\frac{1}{4 \text{length}(\gamma_n)} [Z(\gamma_n, t)] \xrightarrow{n \to \infty} T_{\text{bif}}.
\]

(to be made more precise later)
2. Some ingredients of proof

A main idea is to use Brownian motion instead of geodesic flow.
2. Some ingredients of proof

A main idea is to use Brownian motion instead of geodesic flow.

Note: There is an alternate proof of the existence of $\chi_{\text{geodesic}}$ by Bonatti, Gomez-Mont and Viana.
2. Some ingredients of proof

A main idea is to use Brownian motion instead of geodesic flow.

Note: There is an alternate proof of the existence of $\chi_{\text{geodesic}}$ by Bonatti, Gomez-Mont and Viana.

(loose definition) Brownian motion on $X$ is the data for every $x \in X$ of a probability measure $W_x$ on the set of continuous paths $\omega : [0, \infty) \to X$, satisfying:

1. the law of $\omega(t)$ assuming $\omega(0) = x$ is given by the distribution of heat at time $t$, starting from $\delta_x$ at $t = 0$.
2. Markov property: the law of $\omega(T + t)$ given $\omega(T) = y$ is the law of $\omega(t)$ given $\omega(0) = y$. 
2. Some ingredients of proof

On \( \mathbb{H}^2 \) a typical Brownian path from 0 escapes at speed \( \frac{t}{2} \) to the boundary, i.e. for \( W_0 \) a.e. \( \omega \), \( \lim_{t \to \infty} \frac{1}{t} d_{\mathbb{H}^2}(0, \omega(t)) = \frac{1}{2} \)
(normalization : heat kernel associated to \( \frac{1}{2} \Delta \))
2. Some ingredients of proof

We establish the following

**Proposition**

Given a Brownian path $\omega$ from $\star$, close the Brownian path $\omega|_{[0,t]}$ with some path of length $\leq \text{diam}(X)$ joining $\omega(t)$ to $\star$. Let $\tilde{\omega}_t$ be the corresponding loop.

For $W_\star$ a.e. $\omega$, the limit

$$\chi_{\text{Brown}}(\rho) = \lim_{t \to \infty} \frac{1}{t} \log \| \rho ([\tilde{\omega}_t]) \|$$

exists and does not depend on $\omega$. 

Romain Dujardin  
Bers, Brown and Lyapunov
We establish the following

**Proposition**

Given a Brownian path \( \omega \) from \( \star \), close the Brownian path \( \omega|_{[0,t]} \) with some path of length \( \leq \text{diam}(X) \) joining \( \omega(t) \) to \( \star \). Let \( \tilde{\omega}_t \) be the corresponding loop. For \( \mathcal{W}_\star \) a.e. \( \omega \), the limit

\[
\chi_{\text{Brown}}(\rho) = \lim_{t \to \infty} \frac{1}{t} \log \left\| \rho ([\tilde{\omega}_t]) \right\|
\]

exists and does not depend on \( \omega \).

By definition this is the Lyapunov exponent of \( \rho \) associated to Brownian motion on \( X \).
2. Some ingredients of proof

We establish the following

**Proposition**

Given a Brownian path $\omega$ from $\star$, close the Brownian path $\omega|_{[0,t]}$ with some path of length $\leq \text{diam}(X)$ joining $\omega(t)$ to $\star$. Let $\tilde{\omega}_t$ be the corresponding loop.

For $W_\star$ a.e. $\omega$, the limit

$$\chi_{\text{Brown}}(\rho) = \lim_{t \to \infty} \frac{1}{t} \log \|\rho([\tilde{\omega}_t])\|$$

exists and does not depend on $\omega$.

By definition this is the Lyapunov exponent of $\rho$ associated to Brownian motion on $X$.

Then we get that $\chi_{\text{Brown}} = \frac{1}{2}\chi_{\text{geodesic}}$ and $T_{\text{bif}} = dd^c \chi_{\text{Brown}}$. 
2. Some ingredients of proof

Why Brownian motion is better than geodesics?
2. Some ingredients of proof

Why Brownian motion is better than geodesics?

Answer: Furstenberg’s discretization procedure allows to replace Brownian motion by a discrete random walk on $G$. 
2. Some ingredients of proof

Why Brownian motion is better than geodesics?

Answer: Furstenberg’s discretization procedure allows to replace Brownian motion by a discrete random walk on \( G \).

Let \( \mu \) be a proba measure on \( G \). Define

\[
\chi_\mu(\rho) = \lim_{n \to \infty} \frac{1}{n} \int \log \|\rho(g_1 \cdots g_n)\| \, d\mu(g_1) \cdots d\mu(g_n).
\]
2. Some ingredients of proof

Why Brownian motion is better than geodesics?

Answer: Furstenberg’s discretization procedure allows to replace Brownian motion by a discrete random walk on $G$.

Let $\mu$ be a proba measure on $G$. Define

$$\chi_\mu(\rho) = \lim_{n \to \infty} \frac{1}{n} \int \log \|\rho(g_1 \cdots g_n)\| \, d\mu(g_1) \cdots d\mu(g_n).$$

We proved in a previous work that if $\mu$ satisfies certain moment and non-degeneracy conditions, then $T_{\text{bif},\mu} = dd^c \chi_\mu$ satisfies $\text{Supp}(T_{\text{bif},\mu}) = \text{Bif}$ and some equidistribution properties.
2. Some ingredients of proof

Why Brownian motion is better than geodesics?

Answer: Furstenberg’s discretization procedure allows to replace Brownian motion by a discrete random walk on $G$.

Let $\mu$ be a proba measure on $G$. Define

$$\chi_\mu(\rho) = \lim_{n \to \infty} \frac{1}{n} \int \log \|\rho(g_1 \cdot \cdot \cdot g_n)\| \, d\mu(g_1) \cdot \cdot \cdot d\mu(g_n).$$

We proved in a previous work that if $\mu$ satisfies certain moment and non-degeneracy conditions, then $T_{\text{bif},\mu} = dd^c \chi_\mu$ satisfies $\text{Supp}(T_{\text{bif},\mu}) = \text{Bif}$ and some equidistribution properties.

Here, for $\mu =$Furstenberg’s measure, there exists $\tau$ s.t. for every $\rho$

$$\chi_\mu(\rho) = \tau \chi_{\text{Brown}}(\rho)$$

so we can use our previous work.
2. Some ingredients of proof

What is a random geodesic?
2. Some ingredients of proof

What is a random geodesic?

A typical element in $\Gamma$ relative to $\mu^n$ ($n^{th}$ step of the random walk associated to $\mu$) is primitive and satisfies:

\[
\log \|\gamma\| \simeq \frac{\tau}{4} n \quad \text{and} \quad \log \ell(\gamma) \simeq \frac{\tau}{2} n \quad \text{(translation length)}
\]
2. Some ingredients of proof

What is a random geodesic?

A typical element in $\Gamma$ relative to $\mu^n$ ($n$th step of the random walk associated to $\mu$) is primitive and satisfies:

$$\log \| \gamma \| \simeq \frac{\tau}{4} n \quad \text{and} \quad \log \ell(\gamma) \simeq \frac{\tau}{2} n \quad \text{(translation length)}$$

(get something like a Gaussian measure concentrated around the circle of radius $\simeq \frac{\tau}{2} n$)
2. Some ingredients of proof

Every loop in $X$ is homotopic to a unique closed geodesic so we can project the measure $\mu^n$ to a proba measure $m_n$ on the set of closed geodesics on $X$. This is a Gaussian-like measure concentrated around primitive geodesics of length $\sim \frac{T}{2} n$. 
2. Some ingredients of proof

Every loop in $X$ is homotopic to a unique closed geodesic so we can project the measure $\mu^n$ to a proba measure $m_n$ on the set of closed geodesics on $X$. This is a Gaussian-like measure concentrated around primitive geodesics of length $\simeq \frac{T}{2}n$.

**Theorem**

Put $m = \prod m_n$. For every $t \in \mathbb{C}$, if $(\gamma_n)$ is a $m$-random sequence of closed geodesics, then

$$
\frac{1}{4 \text{length}(\gamma_n)} [Z(\gamma_n, t)] \xrightarrow{n \to \infty} T_{\text{bif}}.
$$
3. Projective structures on $X$
3. Projective structures on $X$

View $X$ as $\Gamma \setminus \mathbb{H}^2$ for some Fuchsian group $\Gamma$, and fix an identification $G \simeq \Gamma$ (Fuchsian representation of $G$).
3. Projective structures on $X$

View $X$ as $\Gamma \setminus \mathbb{H}^2$ for some Fuchsian group $\Gamma$, and fix an identification $G \simeq \Gamma$ (Fuchsian representation of $G$).

A projective structure $\sigma$ on the Riemann surface $X$ is the data of a locally injective map

$$\text{dev}(\sigma) : \mathbb{H}^2 \to \mathbb{P}^1$$

(the developing map of $\sigma$)

satisfying the equivariance property

$$\text{dev} \circ \gamma = \rho(\gamma) \circ \text{dev}$$

for some representation $\rho$ of $\Gamma \simeq G$. By definition $\rho$ is the holonomy map $\text{hol}(\sigma)$. 

Romain Dujardin

Bers, Brown and Lyapunov
3. Projective structures on $X$

- We say $\sigma \sim \sigma'$ if there exists $A \in \text{PSL}(2, \mathbb{C})$ s.t.
  \[ \text{dev}(\sigma') = A \circ \text{dev}(\sigma). \]
  Then \[ \text{hol}(\sigma') = A \circ \text{hol}(\sigma) \circ A^{-1}. \]
We say \( \sigma \sim \sigma' \) if there exists \( A \in \text{PSL}(2, \mathbb{C}) \) s.t. 
\[
\text{dev}(\sigma') = A \circ \text{dev}(\sigma).
\]
Then \( \text{hol}(\sigma') = A \circ \text{hol}(\sigma) \circ A^{-1} \).
3. Projective structures on $X$

- We say $\sigma \sim \sigma'$ if there exists $A \in \text{PSL}(2, \mathbb{C})$ s.t. $\text{dev}(\sigma') = A \circ \text{dev}(\sigma)$. Then $\text{hol}(\sigma') = A \circ \text{hol}(\sigma) \circ A^{-1}$. Hence we get a map

$$P(X) = \text{proj. str./conjugacy} \xrightarrow{\text{hol}} \text{Hom}(G, \text{PSL}(2, \mathbb{C}))/\text{conjugacy}$$

- It turns out that

$$P(X) \simeq \{\text{holom. quad. diff. on } X\} \simeq \mathbb{C}^{3g-3} \text{ (Schwarzian param.)}$$

and hol is a proper holomorphic embedding.
3. Projective structures on $X$

- We say $\sigma \sim \sigma'$ if there exists $A \in \text{PSL}(2, \mathbb{C})$ s.t. 
  $\text{dev}(\sigma') = A \circ \text{dev}(\sigma)$. Then $\text{hol}(\sigma') = A \circ \text{hol}(\sigma) \circ A^{-1}$.

  Hence we get a map

  \[ P(X) = \text{proj. str./conjugacy} \xrightarrow{\text{hol}} \text{Hom}(G, \text{PSL}(2, \mathbb{C}))/\text{conjugacy} \]

- It turns out that

  \[ P(X) \simeq \{ \text{holom. quad. diff. on } X \} \simeq \mathbb{C}^{3g-3} \text{ (Schwarzian param.)} \]

  and hol is a proper holomorphic embedding.

- We thus get a natural holomorphic family of representations (mod. conjugacy) associated to $X$
3. Projective structures on $X$

- We say $\sigma \sim \sigma'$ if there exists $A \in \text{PSL}(2, \mathbb{C})$ s.t. $\text{dev}(\sigma') = A \circ \text{dev}(\sigma)$. Then $\text{hol}(\sigma') = A \circ \text{hol}(\sigma) \circ A^{-1}$.

  Hence we get a map

  $$P(X) = \text{proj. str.}/\text{conjugacy} \overset{\text{hol}}{\rightarrow} \text{Hom}(G, \text{PSL}(2, \mathbb{C}))/\text{conjugacy}$$

- It turns out that

  $$P(X) \sim \{\text{holom. quad. diff. on } X\} \sim \mathbb{C}^{3g-3} \text{ (Schwarzian param.)}$$

  and $\text{hol}$ is a proper holomorphic embedding.

- We thus get a natural holomorphic family of representations (mod. conjugacy) associated to $X$
3. Projective structures on $X$

- We say $\sigma \sim \sigma'$ if there exists $A \in \text{PSL}(2, \mathbb{C})$ s.t. $\text{dev}(\sigma') = A \circ \text{dev}(\sigma)$. Then $\text{hol}(\sigma') = A \circ \text{hol}(\sigma) \circ A^{-1}$.

Hence we get a map

$$P(X) = \text{proj. str.}/\text{conjugacy} \xrightarrow{\text{hol}} \text{Hom}(G, \text{PSL}(2, \mathbb{C}))/\text{conjugacy}$$

- It turns out that

$$P(X) \simeq \{\text{holom. quad. diff. on } X\} \simeq \mathbb{C}^{3g-3} \text{ (Schwarzian param.)}$$

and $\text{hol}$ is a proper holomorphic embedding.

- We thus get a natural holomorphic family of representations (mod. conjugacy) associated to $X$ (notice that $\chi_{\text{Brown}}$ is insensitive to conjugacies so $\chi_{\text{Brown}}(\text{hol}(\sigma))$ is well-defined).
3. Projective structures on $X$

Example

- Standard Fuchsian structure $\sigma_{\text{Fuchs}}$. $\text{hol} = \text{identity}$. 
3. Projective structures on $X$

Example

- Standard Fuchsian structure $\sigma_{\text{Fuchs}}$. $\text{hol} = \text{identity}$.
- Quasi-Fuchsian deformations of $\sigma_{\text{Fuchs}}$. $\text{dev} (\mathbb{H}^2)$ is a quasidisk. The set of such QF deformations is a bounded open set $B(X) \subset P(X) \simeq \mathbb{C}^{3g-3}$, which is the stability component of $\sigma_{\text{Fuchs}}$. “Bers embedding of Teichmüller space”.
3. Projective structures on $X$

**Example**

- Standard Fuchsian structure $\sigma_{\text{Fuchs}}$. $\text{hol} = \text{identity}$.
- Quasi-Fuchsian deformations of $\sigma_{\text{Fuchs}}$. $\text{dev}(\mathbb{H}^2)$ is a quasidisk. The set of such QF deformations is a bounded open set $B(X) \subset P(X) \simeq \mathbb{C}^{3g-3}$, which is the stability component of $\sigma_{\text{Fuchs}}$. “Bers embedding of Teichmüller space”.
- For a general $\sigma$, $\text{hol}(\sigma)$ may be discrete or not. $\text{dev}(\sigma)$ can cover $\mathbb{P}^1$ many times.
3. Projective structures on $X$

**Figure:** Bers slice and stability components (Yasushi Yamashita)
3. The degree of a projective structure
3. The degree of a projective structure

**Definition-Proposition**

Let $\sigma$ be a projective structure on $X$. Fix $z \in \mathbb{P}^1$. Then for any sequence $p_n \in \mathbb{H}^2$ and $r_n \to \infty$ the sequence

$$\frac{1}{\text{Vol} B_{\mathbb{H}^2}(p_n, r_n)} \# \text{dev}^{-1}(z) \cap B_{\mathbb{H}^2}(p_n, r_n)$$

converges to a number $\delta(\sigma)$ independent of the choices: the degree of $\sigma$. 
3. The degree of a projective structure

Definition-Proposition

Let $\sigma$ be a projective structure on $X$. Fix $z \in \mathbb{P}^1$. Then for any sequence $p_n \in \mathbb{H}^2$ and $r_n \to \infty$ the sequence

$$\frac{1}{\text{Vol}B_{\mathbb{H}^2}(p_n, r_n) \# \text{dev}^{-1}(z) \cap B_{\mathbb{H}^2}(p_n, r_n)}$$

converges to a number $\delta(\sigma)$ independent of the choices: the degree of $\sigma$.

Example

For a Fuchsian or QF structure, $\delta = 0$. 
3. The degree of a projective structure

Lyapunov exponent and degree are related by the following formula:
Lyapunov exponent and degree are related by the following formula:

**Theorem**

For every $\sigma \in P(X)$, $\chi_{\text{Brown}}(\sigma) = \frac{1}{4} + \pi \delta(\sigma)$. 
3. The degree of a projective structure

Lyapunov exponent and degree are related by the following formula:

**Theorem**

For every $\sigma \in P(X)$, $\chi_{\text{Brown}}(\sigma) = \frac{1}{4} + \pi \delta(\sigma)$.

This is formally analogous to the Manning-Przytycki formula.
3. The degree of a projective structure

Lyapunov exponent and degree are related by the following formula:

**Theorem**

For every $\sigma \in P(X)$, $\chi_{\text{Brown}}(\sigma) = \frac{1}{4} + \pi \delta(\sigma)$.

This is formally analogous to the Manning-Przytycki formula.

**Corollary**

$\delta$ is a continuous psh function on $P(X)$ and $dd^c \delta = \frac{1}{\pi} T_{\text{bif}}$. 
3. Why $\chi_{\text{Brown}}$ should be constant on the Bers slice?
3. Why $\chi_{\text{Brown}}$ should be constant on the Bers slice?

For a Fuchsian group $\chi_{\text{Brown}} = \frac{1}{4}$: this corresponds to the rate of escape of Brownian motion in $\mathbb{H}^2$. 
For a Fuchsian group $\chi_{\text{Brown}} = \frac{1}{4}$: this corresponds to the rate of escape of Brownian motion in $\mathbb{H}^2$.

For a QF group, consider the Brownian motion starting from some point $0$ in the component uniformizing $X$. The hitting measure on the boundary is the harmonic measure viewed from $0$. 
3. Why $\chi_{\text{Brown}}$ should be constant on the Bers slice?

By Makarov’s theorem $\text{dim}(\text{harmonic measure}) = 1$ so it does not depend on $\rho$. 
3. Why $\chi_{\text{Brown}}$ should be constant on the Bers slice?

By Makarov’s theorem $\dim(\text{harmonic measure}) = 1$ so it does not depend on $\rho$.

Now, for a random walk on $(G, \mu)$, and a discrete rep $\rho$, we have Ledrappier’s formula

$$\dim(\text{stationary measure}) = \frac{\text{entropy}(G, \mu)}{\chi_{\mu}(\rho)}.$$
3. Why $\chi_{\text{Brown}}$ should be constant on the Bers slice?

By Makarov’s theorem $\dim(\text{harmonic measure}) = 1$ so it does not depend on $\rho$.

Now, for a random walk on $(G, \mu)$, and a discrete rep $\rho$, we have Ledrappier’s formula

$$\dim(\text{stationary measure}) = \frac{\text{entropy}(G, \mu)}{\chi_{\mu}(\rho)}.$$ 

Applying this to Furstenberg’s discretization measure we conclude that $\chi_{\text{Brown}}$ is constant on the Bers slice.
3. Why $\chi_{\text{Brown}}$ should be constant on the Bers slice?

By Makarov’s theorem $\dim(\text{harmonic measure}) = 1$ so it does not depend on $\rho$.

Now, for a random walk on $(G, \mu)$, and a discrete rep $\rho$, we have Ledrappier’s formula

$$\dim(\text{stationary measure}) = \frac{\text{entropy}(G, \mu)}{\chi_\mu(\rho)}.$$ 

Applying this to Furstenberg’s discretization measure we conclude that $\chi_{\text{Brown}}$ is constant on the Bers slice.

Conversely the degree formula gives an independent proof of Makarov’s theorem for QF groups. This is similar to the case of polynomials with connected Julia sets.
3. A new convexity property of the Bers embedding
3. A new convexity property of the Bers embedding

**Theorem**

\[ B(X) \text{ is polynomially convex.} \]
3. A new convexity property of the Bers embedding

**Theorem**

\( B(X) \) is polynomially convex.

Equivalently, \( B(X) \) is defined by countably many polynomial equations of the form \( \{|P_\alpha| \leq 1\} \).
3. A new convexity property of the Bers embedding

**Theorem**

$B(X)$ is polynomially convex.

Equivalently, $B(X)$ is defined by countably many polynomial equations of the form $\{|P_\alpha| \leq 1\}$.

**Corollary (Shiga)**

$B(X)$ is polynomially convex.
Theorem
\( B(X) \) is polynomially convex.

Equivalently, \( B(X) \) is defined by countably many polynomial equations of the form \( \{ |P_\alpha| \leq 1 \} \).

Corollary (Shiga)
\( B(X) \) is polynomially convex.

These two notions of polynomial convexity differ.
3. A new convexity property of the Bers embedding

**Theorem**

$B(X)$ is polynomially convex.

Equivalently, $B(X)$ is defined by countably many polynomial equations of the form $\{|P_\alpha| \leq 1\}$.

**Corollary (Shiga)**

$B(X)$ is polynomially convex.

**Proof of the theorem**: by general properties of psh functions, what we need to show: $B(X)$ is a connected component of $\{\delta = 0\}$. 
Theorem

\( B(X) \) is polynomially convex.

Equivalently, \( B(X) \) is defined by countably many polynomial equations of the form \( \{ |P_\alpha| \leq 1 \} \).

Corollary (Shiga)

\( B(X) \) is polynomially convex.

Proof of the theorem: by general properties of psh functions, what we need to show: \( B(X) \) is a connected component of \( \{ \delta = 0 \} \).

Note: \( \{ \delta = 0 \} \neq B(X) \).
3. Proof of the degree formula/existence of the degree

Based on the study of the suspension of $\rho$:

$$X_\rho = \mathbb{H}^2 \times \mathbb{P}^1 \big/ \Gamma \otimes \rho(G) \ldots$$

....to be continued on whiteboard.