Affine Deligne-Lusztig Varieties, Newton Polygons, and Quantum Schubert Calculus

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Affine Deligne-Lusztig Varieties

- Let $F = \overline{\mathbb{F}}_q((t)) \supset \mathcal{O} = \overline{\mathbb{F}}_q[[t]]$. The Frobenius $\sigma$ acts on coefficients by $\sigma(\sum a_i t^i) = \sum a_i^q t^i$.
- Let $G = SL_n(F) \supset B =$ upper triangulars $\supset T =$ diagonals.

- The Iwahori subgroup $I = \begin{pmatrix} O \times & \ldots & O \\ \vdots & \ddots & \vdots \\ tO & \ldots & O^{\times} \end{pmatrix}$.

- The Weyl group $W = S_n$ and the affine Weyl group $\tilde{W} = \tilde{S}_n$.
- The affine Bruhat decomposition says that $G = \bigsqcup_{x \in \tilde{W}} IxI$.

- The affine Deligne-Lusztig variety associated to a fixed $x \in \tilde{W}$ and $b \in G$ is defined to be
  \[ X_x(b) = \{ g \in G/I \mid g^{-1}b\sigma(g) \in IxI \}. \]

- Motivated by study of Shimura varieties; the geometry of ADLVs is governed by subtle combinatorics of $\tilde{W}$. 
- Fix an element $b \in G$. We can associate to $b$ a *Newton polygon* $\nu(b)$ in the following manner:

  - Compute a $\sigma$-twisted version of the characteristic poly
  - Plot the valuations of each coefficient in this polynomial
  - Take the upper convex hull

- **Theorem** (Kottwitz): The set $\mathcal{N}(G) = \{\nu(g) \mid g \in G\}$ indexes $\sigma$-twisted conjugacy classes in $G$.

- To study affine Deligne-Lusztig varieties, we can instead fix an $x \in \tilde{W}$ and study the set

  $$\mathcal{N}(G)_x = \{\nu(g) \mid g \in IxI\}.$$ 

- The set $\mathcal{N}(G)_x$ is a *partially ordered set*, and it contains a unique maximal and minimal element.

- Many interesting open questions about this poset remain!
- The *quantum cohomology ring* of the complex complete flag variety $X = SL_n(\mathbb{C})/B$ equals

$$QH^*(X) = H^*(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \ldots, q_{n-1}].$$

- As a $\mathbb{Z}[q_1, \ldots, q_{n-1}]$-module, $QH^*(X)$ has a basis of *Schubert classes* $\sigma_w$ where $w \in W$.

- The main problem in modern quantum Schubert calculus is to explicitly compute the products $\sigma_u \ast \sigma_v = \sum_{w,d} c_{u,v}^{w,d} q^d \sigma_w$ by finding non-recursive, positive combinatorial formulas for the coefficients $c_{u,v}^{w,d}$ and monomials $q^d$.

- **The Curious Connection:** The unique maximal element in $\mathcal{N}(G)_x$ and the unique minimal monomial $q^d$ in these quantum products are governed by precisely the same combinatorics!
- The combinatorics of the Bruhat-Tits building and, more specifically, of the quantum Bruhat graph is a common tool. Here is the quantum Bruhat graph for $S_3$:

![Quantum Bruhat Graph](image)

- **The Key Point:** Paths in the quantum Bruhat graph correspond to saturated chains in affine Bruhat order.
- This observation is also fundamental in the work of Schilling et al on affine crystals and Macdonald polynomials.