$W = \text{Faloves} \,$

$0^* = \{ \text{hexagons} \}$  \hspace{1cm} $W_0 = \{ \text{Faloves on the}\}$

$s_0, s_1, \ldots, s_n$ are reflections in $0^*$, $0^*$, $\ldots$, $0^*$ the walls of $I$; $I_0 = \gamma_0^* + \gamma_1^*$

$\mu$ translate of 1 to the $\mu$-hexagon,

$\langle \mu, v^* \rangle = \text{distance from } \mu \text{ to } 0^*$

$L(w) = \text{length of a reduced word for } w$

$w = s_i; \ldots; s_k \text{ (min length path to } w)$

$W_0 = \text{longest element of } W_0$

The periodic orientation has the north star on the positive side of all hyperplanes.
Hecke algebras

The double affine braid group $\tilde{B}$ is generated by $T_0, T_1, \ldots, T_n, \quad X^\mu, \mu \in \mathbb{Z}^n, \quad q$

with $q \in \mathbb{Z}(B), \quad X^\mu X^\nu = X^{\mu+\nu}$

$\prod_{i \neq j} T_i T_j = \prod_{i \neq j} T_j T_i$

$T_i X^\mu = \begin{cases} X^{\mu T_i}, & \text{if } \langle \mu, e^\cdot T_i \rangle = 0, \\ X^{\mu T_i^{-1}}, & \text{if } \langle \mu, e^\cdot T_i \rangle = 1. \end{cases}$

The double affine Hecke algebra $\tilde{H}$ is $\mathbb{C}[B]$ with

$T_i = (t^{E_i} - t^{-E_i}) T_i + 1, \quad \text{for } i = 0, 1, \ldots, n.$

$\tilde{H}$ has a $\mathbb{C}[t^E, t^{-E}]$-basis

$\{q^k X^\mu T_w Y^\lambda | k \in \mathbb{Z}, \mu \in \mathbb{Z}^n, w \in W_0, \lambda \in \mathbb{Z}^n \}$

where $T_w = T_{i_1} \cdots T_{i_n}$ if $w = s_{i_1} \cdots s_{i_n}$ is reduced and $I_{\lambda} = s_{i_1} \cdots s_{i_2}$ is reduced then

$Y^\lambda = T_{i_1} \cdots T_{i_2}$

where $e_k = \begin{cases} 1, & \text{if the } k\text{th step of } s_{i_1} \cdots s_{i_2} \text{ is } +1 \\ -1, & \text{if the } k\text{th step of } s_{i_1} \cdots s_{i_2} \text{ is } -1 \end{cases}$

Remark: $Y^\lambda Y_0^\nu = Y^{\lambda + \nu}$
Intertwiners and Macdonald polynomials

\[ t_i = T_i + \frac{t^2(1-t)}{1-y^{-t}}, \quad \text{for } i = 0, 1, \ldots, n. \]

and \( t_n = s_{i_1} \cdots s_{i_2} \) if \( w = s_{i_1} \cdots s_{i_2} \) reduced. Then
\[ t_w y^w = y^{\pi w} t_w, \quad \text{for } w \in W \text{ and } \pi \in \mathbb{Y}. \]

The polynomial representation of \( \mathfrak{h} \) is
\[ \mathfrak{h} = \text{CE}(X^1) = \text{Ind}_H^G(X^1) \text{ with basis } \{ q^k X^\mu | k \in \mathbb{Z}, \mu \in \mathfrak{g}^* \} \]
where \( T_i^\mu = t_i^\mu \) \( \forall i = 0, 1, \ldots, n. \)

Let \( \mu = s_{i_1} \cdots s_{i_2} \) be a minimal length path to the \( \mu \)-hexagon.

The nonsymmetric and symmetric Macdonald polynomials are

\[ E_\mu = E_\mu(q,t) = t_\mu^\mu \quad \text{and} \quad P_\mu = P_\mu(q,t) = t_\mu^\mu E_\mu. \]

where \( t_\mu = \sum_{w \in W_\mu} t^\mu_{w} \)

so that \( t_0 t_i = t_i t_0 \) for \( i = 1, 2, \ldots, n. \)

Remarks: \( E_\mu \) are simultaneous eigenvectors of \( Y^w \) on \( \mathfrak{h}^H = \text{CE}(X^1). \)

\( P_\mu \) are simultaneous eigenvectors of \( f(X) \in \text{CE}(X^1)^W \) on \( \mathfrak{h}^H = \text{CE}(X^1)^W. \)

\( P_{\mu}(0,t) = \text{Hall-Littlewood polynomial} \quad \text{and} \quad \text{Macdonald spherical function} \)

\( P_{\mu}(1,t) = \text{Hall-Littlewood polynomial} \quad \text{and} \quad \text{Macdonald spherical function} \)

\( P_{\mu}(0,0) = s_\mu = \text{Schur function} \quad \text{and} \quad \text{Weyl character}. \)
Path formulas

A step of type \( j \) is

\[
\begin{align*}
&\vec{v} \quad | \quad \vec{v}_i \quad \text{or} \quad \vec{v} \quad | \quad \vec{v}_i \\
&\vec{v} \quad | \quad \vec{v}_i \quad \text{or} \quad \vec{v} \quad | \quad \vec{v}_i \quad \text{or} \quad \vec{v} \quad | \quad \vec{v}_i
\end{align*}
\]

Fix \( \vec{s} = s_i \ldots s_k \) a minimal length path to the \( \mu \)-hexagon.

\[
P(\vec{s}, v) = \{ \text{paths of type } (i, \ldots, s_k) \text{ beginning at } v \}
\]

and \( p \) = end alcove of \( p = t\nu t_1 g(p) \), with \( \nu t_1 g(p) \in W_0 \\
\]

\[
f^+ (p) = \{ k | \text{the step of } p \text{ is } \vec{v}^+ \}
\]

\[
f^- (p) = \{ k | \text{the step of } p \text{ is } \vec{v}^- \}
\]

\[
WT(p) = \left( \prod \frac{x^{e^+ (1-t)}_{\vec{y}}} {x^{e^- (1-t)}_{\vec{y}}} \right) \left( \prod \frac{x^{e^+ (1-t)}_{\vec{y}}} {x^{e^- (1-t)}_{\vec{y}}} \right)
\]

where \( \vec{y}, \ldots, \vec{y} \) are the hyperplanes crossed by \( s_i \ldots s_k \), and

\[
y_{x_1 x_2} = q \langle \vec{y}, p \rangle
\]

Theorem (Yip-Ram)

\[
E_\mu = \sum_{p \in P(\vec{s}, t)} x^{wt(p)} t^{\ell(g(p))} WT(p)
\]

and

\[
P_\mu = \sum_{v \in W_0} \sum_{p \in P(\vec{s}, v)} t^{\ell(g(p))} x^{wt(p)} t^{\ell(g(p))} WT(p)
\]
Theorem. In $H$, 

\[ T \nu \kappa^2 = \sum_{p \in P(\mu, v)} n(p) \frac{\prod \frac{t^{\frac{1}{2}}(1-t)}{1-t^{v^2}}}{\prod \frac{t^{\frac{1}{2}}(1-t)}{1-t^{v^2}}} \]