Period integrals of automorphic forms

The purpose of the talk is to give a bird's view of some aspects of period integrals and their relation with other topics of interest of automorphic forms and automorphic reps.

Classical Example:

\[ E(z,s) = \sum_{m \neq 0} \frac{\text{Im}(z)^s}{|mz-1|^s} \]

Normalized Real analytic Eisenstein series on \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \)

\[ E(z,s) = \sum (\xi) L(5, \chi_\xi) = I_Q(\xi) \]

Waldspurger (86'): If Hecke-Maass cusp form on \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \)

\[ \|f\|^2, |f(x)|^2 \sim L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \chi_\xi) = L(\frac{1}{2}, \text{bc}(f)) \]

be: \( \text{Aut}(\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}) \to \text{Aut} \mathcal{L}_2(\mathbb{Z}) \]

In this context of complex extension 1st defined by Shintani (79') and generalized by Langlands (80') and Arthur Clozel (81')

Both formulas generalize to weighted sums of Heegner points of a fixed negative discriminant (replacing -4).
Adelic version:

\[ \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \approx \mathbb{R}(\mathbb{A}) \backslash \text{GL}_2(\mathbb{A}) \big/ (\mathcal{O} \times \prod_p \text{GL}_2(\mathbb{A}_p)) \]

\[ \text{f- modular } \phi \rightarrow \text{ f- } \text{GL}_2(\mathbb{A}_0) \rightarrow \prod_p \text{GL}_2(\mathbb{A}_p) \]

\[ \exists \text{ forms } \Gamma = \text{GL}_2, \Gamma = \{a \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \} \text{ s.t. } \Gamma(\mathbb{Q}) \approx \mathbb{Q}[i] \times \mathbb{Q}[i] \]

\[ \int_{\mathbb{A}_0 \backslash \mathbb{Q}[i]} d \tau = \mathcal{P}_{\Gamma}(\phi) = \text{ Period integral } \]

**General Framework**

\[ F - \# \text{ field, } \mathbb{A} = \mathbb{A}_F \]

\[ G \text{ - reductive over } F \]

\[ H - \text{ closed subgp. } \]

\[ \phi: \mathcal{G}(\mathbb{A}) \rightarrow \mathbb{C} \text{ cont. aut. functin} \]

\[ P_H(\phi) = \int_{H(\mathbb{A}) \backslash \mathcal{G}(\mathbb{A})} \chi(h) \phi(h) \, dh \]

**Rule:** \( H - \text{ semi-simple, } \phi \text{- cusp, } P_H \text{ is abs. convergent (Ash-Shimura-Ramis)} \)

**Def.:** An aut. rep. \( \pi \) is \( \mathcal{H} \)-distinguished if \( \frac{P_H}{\mathcal{H}} \neq 0 \).

(Whittaker functional = \( P_{\pi, y} \), \( N \)-mon. wip. \( \pi \)- mon. deg. \( y \) )

We already saw a relation of Period integrals to special \( L \)-values.

Often, distinction also characterizes the image of a functorial lifting. Periods are then related to Langlands functoriality.
Example: Unitary Periods & quad. base-change

\[ E/F - \text{ quad. ext. of } \# \text{ fields} \]
\[ \chi \in \text{ quad. char. of } \Gamma(\mathcal{H}) \]
\[ \psi_{E/F} \text{ the quad. char. of } \Gamma(\mathcal{H}_E) \]

\[ \psi_{E/F} : \text{ Aut. Reps. } \text{GL}_n(\mathcal{H}_F) \longrightarrow \text{ Aut. Reps. } \text{GL}_n(\mathcal{H}_E) \]
\[ \psi_{E/F} : \pi' \longrightarrow \pi = \psi_{E/F}(\pi') \]
\[ L(s, \pi) = L(s, \pi') L(s, \pi' \otimes 2) \]

Characterization of the image: \( \pi \) is cusp. rep. of \( \text{GL}_n(\mathcal{H}_E) \)
\( \pi \) is Galois invariant \( \iff \pi = \psi_{E/F}(\pi') \) is \( \text{H-dist.} \) where \( \psi_{E/F} \) is base-change \( \text{H=U}_n(E/F) = q\text{-split unitary gp.} \)

Arthur - Clozel (89') Jacquet (2010)

Relation with L-values: \( \pi = \psi_{E/F}(\pi') \in \mathcal{D} \)
\[ \| \phi \|_{\psi_{E/F}}^2 \frac{|D_{U_n(E/F)}(\phi)|^2}{L(1, \pi' \otimes \pi' \otimes 2)} \approx \frac{L(1, \pi' \otimes \pi' \otimes 2)}{L(1, \pi' \otimes \text{Ad})} \]

Unlike Waldspurger's formula here the \( L \)-values are at the edge of the critical strip and we know more about them.

When considering compact unitary periods they become sums of points evaluating. Thus known lower bounds to the RHS indicate that aut. forms that are base-change lifts have

large \( L \)-norms. This is compatible with a conjecture of Sarnak on \( L \)-norms of aut. forms.

To obtain these results Jacquet developed, over many years, his Relative Trace Formulae. At the heart of the relation between Unitary Periods and base-change is a geometric relation.
between double cosets.

\[ N_\psi \backslash Gln(F) / N_\nu(F) \cong N_\psi \backslash Gln(E) / U_n(E/F) \]

**Another Setting: Orthogonal Periods** \((G, H) = (Gln, O_n)\)

\[ N_\psi \backslash Gln / N_\nu \cong N_\psi \backslash Gln / O_n \]

\[ \text{Aut. Repr.}(Gln) \xrightarrow{\text{Flicker-Kedlaya}} \text{Aut. Repr.}(Gln) \]

(Jacquet's conjecture: Orthogonal Periods on \(Gln\) are related to Whittaker vectors on \(Gln\) via F-K.)

Hence by Brubaker-Bump-Friedberg to \(A_{n}^{\pm}\) WMBS.

Evidence for \(GL_1(Z)\) Eisenstein Series: Chinta-Owned, Li-Mei Lim

Recently: Vief, Conig proved the Fundamental Lemma for the relevant RTF.

Given the conjecture we could hope to understand orth. period of Eig. series via WMBS. But for cusp forms maybe we could relate orth. periods to metaplectic Whittaker, but we understand neither.
Def: \((G,H)\) is a vanishing pair if

no cusp. rep. of \(G\) is \(H\)-dist.

Example: \((G,G)\) is a vanishing pair since \(P_g(y)=(y,1), \perp L^2_{cusp}.

The following are vanishing pairs:

Jacquet-Rallis \(GL_{2n}, Sp_n\)

Friedberg-J \(GL_{2m}, GL_n, GL_n\)

What about the non-cuspidal spectrum?

\[ L^2(G(F) \backslash G(A)) = L^2_{disc}(G) \oplus L^2_{cont}(G) \]
\[ \quad \quad \quad \quad L^2_{cusp}(G) \oplus L^2_{res}(G) \]

Example: \(L^2_{disc}(GL_n) = \bigoplus U(\sigma, k)\)

\[ n = km, \sigma - irr. cusp. \text{ mult. rep. of } GL_n(A) \]

\[ \bigoplus U(\sigma, k) = L^2 \mathbb{Q} \text{ Ind}_G^{GL_n} (\sigma 1 \otimes \cdots \otimes \sigma 1 \otimes \cdots) \]

Theorem (Symplectic Periods) \((G,H) = (GL_n, Sp_n)\)

1. \(P_H \mid U(\sigma, k)\) is abs. conv.
2. \(U(\sigma, k)\) is \(H\)-dist. iff \(k\) is even.
\[ \text{Let } \mathcal{L}(G) = \bigoplus_{\lambda \in \mathcal{L}(G)} \text{Ind} (\mathbb{C} \lambda) \] 
\[ \text{discrete series on } L \text{ proper Levi } \] 
\[ \Gamma = \bigotimes_i \mathbb{U}(\mathfrak{g}_i) \] 
\[ \text{associate to } \] 
\[ \text{Eis. series } \] 
\[ \mathcal{E}(\varpi_i, \gamma_i) \] 

Then (Yamama) \[ (G, H) = (GL_n, Sp_n) \] 

\[ \mathcal{P}_H (E(\Gamma, \varpi)) \text{ is abs. conv. for } \varpi \in \mathcal{M}^+ \] 

\[ \mathcal{P}_H \mathcal{I}(\mathcal{L}, \varpi) \neq 0 \text{ iff } \Gamma = \bigotimes_i \mathbb{U}(\mathfrak{g}_i, \varpi_i) \] 

The case \[ (G, H) = (Sp_{2n}, Sp_n \times Sp_n) \] 

\[ H \text{-distinction is related to the descent construction of } \] 

Geezberg - Rallis - Sondry. \[ \text{Aut}(Sp_{2n}) \xrightarrow{FJ} \text{Aut}(Sp_n) \] 

Let \[ \tau = \tau_1 \otimes \cdots \otimes \tau_n \] \[ \text{irr. cusp. aut. rep. of } \text{GL}_n \times \cdots \times \text{GL}_n, \text{ s.t.} \] 

\[ (n \text{-Friedberg}) \] 
\[ n \vdash (\mathfrak{g}_1, \mathfrak{g}_2) \text{-dist. } \] 
\[ n \vdash (\mathfrak{g}_i, \mathfrak{g}_j) \] 
\[ n \vdash (\mathfrak{g}, \mathfrak{g}) \] 
\[ n_1 \neq n \] 

Assume \[ n_1 = n_2 + \cdots + n_k \] 

Let \[ M \text{ be the Levi of } Sp_{2n} = \text{GL}_n \times \cdots \times \text{GL}_n \] 

considers \[ \tau \] as a cusp. rep. of \[ M \] 

\[ \text{Ind}_{Sp_{2n}}^{Sp_n} (\tau_1 \otimes \cdots \otimes \tau_n) \xrightarrow{\text{Res}_{\mathfrak{g} = \mathfrak{g}_1 \cdots \mathfrak{g}_n}} \mathcal{E}(\tau) \text{ - irr. cusp. rep. of } Sp_n \] 

\[ (G \text{-E-S}) \] 

Let \[ \phi(\tau) = FJ (\mathcal{E}(\tau)) \text{ - irr. cusp. gen. aut. of } Sp_n \]
The (Laplace - $\mathcal{O}$): $(G, \cdot H) = (S_p, S_p \times S_p)$

$P_H$ is abs. conv. in $\mathcal{E}$ and is $H$-dist.

With Laplace we have further results on the distinguished spectrum. We hope to apply them with some to show that the descent construction gives all the cuspidal generic spectrum of $S_p$. 
