Characterizing extremal limits

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Rademacher Problem

- $g(n, m) := \min\{\#K_3(G) : v(G) = n,\ e(G) = m\}$
- Mantel 1906, Turán'41: $\max\{m : g(n, m) = 0\} = \lfloor \frac{n^2}{4} \rfloor$
- Rademacher'41: $g(n, \lfloor \frac{n^2}{4} \rfloor + 1) = \lfloor \frac{n}{2} \rfloor$

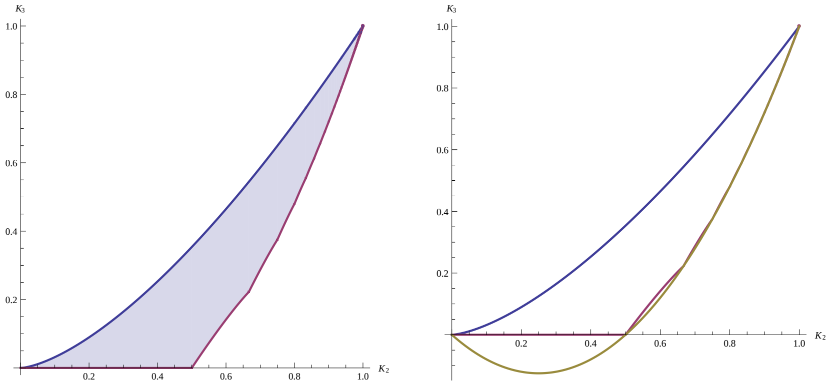
Just Above the Turán Function

- Erdős'55: $m \le \lfloor \frac{n^2}{4} \rfloor + 3$
- Erdős'62: $m \le \lfloor \frac{n^2}{4} \rfloor + \varepsilon n$
- Erdős'55: Is $g(n, \lfloor \frac{n^2}{4} \rfloor + q) = q \cdot \lfloor \frac{n}{2} \rfloor$ for $q < n/2$?
  - $K_{k,k} + q$ edges versus $K_{k+1,k-1} + (q+1)$ edges
- Lovász-Simonovits'75: Yes
- Lovász-Simonovits'83: $m \le \lfloor \frac{n^2}{4} \rfloor + \varepsilon n^2$

Asymptotic Version

- $g(a) := \lim_{n\to\infty} \frac{g(n,a\binom{n}{2})}{\binom{n}{3}}$
- Upper bound: $K_{cn,\ldots,cn,(1-tc)n}$
- Moon-Moser'62, Nordhaus-Stewart'62 (Goodman'59): $g(a) \geq 2a^2 - a$
- Bollobás'76: better lower bound
- Fisher'89: $g(a)$ for $\frac{1}{2} \leq a \leq \frac{2}{3}$
- Razborov'08: $g(a)$ for all a
- No stability
  - H_n^a: modify the last two parts of $K_{cn,\ldots,cn,(1-tc)n}$
- P.-Razborov $\geq$'15: $\forall$ almost extremal G_n is $o(n^2)$-close to some H_n^a

Possible Edge/Triangle Densities



- Upper bound: Kruskal’63, Katona’66

Limit Object

- Subgraph density

$$p(F,G) = \textbf{Prob}\{\, G[\,\text{random } v(F)\text{-set}\,] \cong F\,\}$$

- $\mathcal{F}_0 = \{\text{finite graphs}\}$
- (G_n) converges if $v(G_n) \to \infty$ and

$$\forall\ F \in \mathcal{F}_0 \quad \exists \lim_{n\to\infty} p(F, G_n) =: \phi(F)$$

- $\mathrm{LIM} = \{\text{all such } \phi\} \subseteq [0,1]^{\mathcal{F}_0}$
- $g(a) = \inf\{\phi(K_3) : \phi \in \mathrm{LIM},\ \phi(K_2) = a\}$

Razborov's Flag Algebra $\mathcal{A}^0$

- $\phi \in \mathrm{LIM} \subseteq [0,1]^{\mathcal{F}_0}$
- $\mathcal{F}^0 = \{\text{unlabeled graphs}\}$
- $\mathbb{R}\mathcal{F}^0 := \{\text{quantum graphs}\} = \{\sum \alpha_i F_i\}$
- Linearity: $\phi : \mathbb{R}\mathcal{F}^0 \to \mathbb{R}$
- $\mathcal{A}^0 := \mathbb{R}\mathcal{F}^0 / \langle \text{linear relations that always hold} \rangle$
- $\phi(F_1)\phi(F_2) = \sum c_H \phi(H)$
- Define: $F_1 \cdot F_2 := \sum_H c_H H$
- $\phi : \mathcal{A}^0 \to \mathbb{R}$ is algebra homomorphism

Positive Homomorphisms

- $\phi \in \mathrm{Hom}(\mathcal{A}^0, \mathbb{R})$ is positive if $\forall\, F \in \mathcal{F}^0\ \phi(F) \geq 0$
- $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) = \{$positive homomorphisms$\}$
- Lovász-Szegedy'06, Razborov'07:
  $\mathrm{LIM} = \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$
- $\supseteq$: Let $\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})$
  - $\sum_{|F|=n} \phi(F) = 1$
  - Distribution on $\mathcal{F}^0_n$
  - **Prob**[random $G_n \to \phi$] $= 1$
  - $\phi \in \mathrm{LIM}$
- Write $\sum \alpha_i F_i \geq 0$ if
  - $\forall \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R})\ \ \sum \alpha_i \phi(F_i) \geq 0$
  - Equivalently: $\forall (G_n)\ \ \liminf \sum \alpha_i p(F_i, G_n) \geq 0$

Limit version of the problem

- $g(a) = \min\{\phi(K_3) : \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}),\ \phi(K_2) = a\}$
- Characterise all extremal ϕ:

$$\{\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) : \phi(K_3) = g(\phi(K_2))\} \ = ?$$

- Alternatively: work with graphons. E.g.

$$g(a) = \min\{t(K_3, W) : \text{graphon } W \text{ with } t(K_2, W) = a\}$$

Razborov's Proof for $a \in [\frac{1}{2}, \frac{2}{3}]$

- $h(a) =$ conjectured value
- $\mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) \subseteq [0,1]^{\mathcal{F}}$ is closed $\Rightarrow$ compact
- $f(\phi) := \phi(K_3) - h(\phi(K_2))$ is continuous
- $\exists$ ϕ_0 that minimises f on $\{\phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) : \frac{1}{2} \leq \phi(K_2) \leq \frac{2}{3}\}$
- $a := \phi_0(K_2)$
- $c : e(K_{cn,cn,(1-2c)n}) \approx a\binom{n}{2}$

Goodman bound

- Goodman bound: $K_3 - K_2(2K_2 - 1) \geq 0$
  - Cauchy-Schwarz: $[\![K_2^1 \cdot K_2^1]\!]_1 \geq K_2 \cdot K_2$
  - $[\![K_2^1 \cdot K_2^1]\!]_1 = \frac{1}{2} K_3 + \frac{1}{2} K_2 - \frac{1}{6} \bar{P}_3 \leq \frac{1}{2} K_3 + \frac{1}{2} K_2$
- Assume $\frac{1}{2} < a < \frac{2}{3}$
  - Otherwise done by the Goodman bound
- h is differentiable at a

At Most *cn* Triangles per Edge

- Pick $G_n \to \phi_0$
  - Rate of growth: $g(n, m+1) - g(n, m) \approx cn$
  - $\approx cn$ triangles per new edge
  - G_n has $\lessapprox cn$ triangles on almost every edge
- Flag algebra statement

$$\phi_{\mathbf{0}}^{\mathbf{E}}(K_3^E) \le c \quad \text{a.s.}$$

- Informal explanation:
  - $\phi_{\mathbf{0}}^{\mathbf{E}}$: Two random adjacent roots $\boldsymbol{x_1}, \boldsymbol{x_2}$ in G_n
  - K_3^E: Density of rooted triangles

Flag Algebra $\mathcal{A}^E$

- $E := (K_2, 2 \text{ roots})$
- $\mathcal{F}^E := \{(F, x_1, x_2) : F \in \mathcal{F}^0, x_1 \sim x_2\}$
- $p(F, G)$: root-preserving induced density
- $G_n \in \mathcal{F}^E$ converges if $\forall F \in \mathcal{F}^E$ $p^E(F, G_n) \to \phi^E(F)$
- $\phi^E : \mathbb{R}\mathcal{F}^E \to \mathbb{R}$
- $\mathcal{A}^E := \left(\mathbb{R}\mathcal{F}^E / \langle \text{trivial relations} \rangle, \text{multiplication} \right)$
- Razborov'07: $\{\text{limits } \phi^E\} = \mathrm{Hom}^+(\mathcal{A}^E, \mathbb{R})$
- Random homomorphism $\phi_{\mathbf{0}}^{\boldsymbol{E}}(K_3^E)$:
  - $G_n \to \phi$
  - $(G_n, [\text{random } x_1 \sim x_2]) \in \mathcal{M}(\mathcal{F}^E)$
  - Weak limit

Vertex Removal

- Remove $x \in V(G_n)$:
  - $\partial\, p(K_2, G_n)$:
    - Remove edges: $-d(x)/\binom{n}{2}$
    - Remove isolated x: $\times \binom{n}{2}/\binom{n-1}{2} = 1 + \frac{2}{n} + ...$
    - Total change: $-K_2^1(x)/\binom{n}{2} + a\,\frac{2}{n} + ...$
  - $\partial\, p(K_3, G_n) = -K_3^1(x)/\binom{n}{3} + \phi_0(K_3)\,\frac{3}{n} + ...$
- Expect: $\partial p(K_3) \gtrapprox h'(a)\,\partial p(K_2)$
- Cloning x: signs change
- Approximate equality for almost all x
- Flag algebra statement:

$$-3!\,\phi_{\mathbf{0}}^{\mathbf{1}}(K_3^1) + 3\phi_0(K_3) = 3c\left(-2\phi_{\mathbf{0}}^{\mathbf{1}}(K_2^1) + 2a\right) \quad a.s.$$

Finishing line

- Recall: A.s.
  - $-3!\,\phi_{\mathbf{0}}^{\mathbf{1}}(K_3^1) + 3\phi_0(K_3) = 3c\left(-2\phi_{\mathbf{0}}^{\mathbf{1}}(K_2^1) + 2a\right)$
  - $\phi_{\mathbf{0}}^{\boldsymbol{E}}(K_3^E) \leq c$
- Average?
  - $0 = 0$ ☹
  - Slack ☹
- Multiply by K_2^1 & $\overline{P}_3^E$ and then average!
- Calculations give

$$\phi_0(K_3) \geq \frac{3ac(2a-1) + \phi_0(K_4) + \frac{1}{4}\phi_0(\overline{K}_{1,3})}{3c + 3a - 2}$$

- $\phi_0(K_4) \geq 0$ & $\phi_0(\overline{K}_{1,3}) \geq 0 \;\Rightarrow\; \phi_0(K_3) \geq h(a)$ ☺

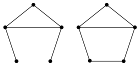
Extremal Limits

- Extremal limit: limits of almost extremal graphs
- Equivalently: $\{\, \phi \in \mathrm{Hom}^+(\mathcal{A}^0, \mathbb{R}) : \phi(K_3) = g(\phi(K_2)) \,\}$
- P.-Razborov $\geq$'15: $\{$extremal limits$\} = \{$limits of H_n^a's$\}$
- Implies the discrete theorem
  - Pick a counterexample (G_n)
  - Subsequence convergent to some ϕ
  - $H_n^a \to \phi$
  - $\delta_\square(G_n, H_n^a) \to 0$
    - Overlay $V(G_n) = V(H_n^a) = V_1 \cup \cdots \cup V_{t-1} \cup U$
    - $G[V_i, \overline{V_i}]$ almost complete
    - $G[V_i]$ almost empty
    - $G[U]$ has $o(n^3)$ triangles

Structure of Extremal ϕ_0

- Assume $\phi_0(K_3) = h(a)$ $(= g(a))$ with $\frac{1}{2} \leq a \leq \frac{2}{3}$
- If $a \in \{\frac{1}{2}, \frac{2}{3}\}$:
  - Goodman's bound is sharp
  - $\phi_0(\bar{P}_3) = 0$
  - Complete partite
  - Cauchy-Schwarz $\Rightarrow$ regular
  - ϕ_0 is the balanced k-partite limit, done! ☺
- Suppose $a \in (\frac{1}{2}, \frac{2}{3})$
- Density of K_4 and $\overline{K}_{1,3}$ is 0
- If $\phi_0(\overline{P}_3) = 0$,
  - Complete partite
  - K_4-free $\Rightarrow$ at most 3 parts $\Rightarrow$ done! ☺

Case 2: $\phi_0(\overline{P}_3) > 0$

- Special graphs F_1 and F_2: 
- Claim: $\phi_0(F_1) = \phi_0(F_2) = 0$
- Claim: Exist many $\overline{P}_3$'s st
  - $|A| = \Omega(n)$: vertices sending 3 edges to it
  - $|B| = \Omega(n)$: vertices sending $\leq$ 2 edges to it
- Non-edge across $\rightarrow$ a copy of F_1, F_2, or $\overline{K}_{1,3}$
- $G_n[A, B]$ is almost complete
- K_4-freeness + calculations $\Rightarrow$ ☺

Clique Minimisation Problem

- Open: Exact result for K_3
- Nikiforov'11: Asymptotic solution for K_4
- Reiher $\geq$'15: Asymptotic solution for K_r
- Open: Structure & exact result

General Graphs

- Colour critical: $\chi(F) = r + 1$ & $\chi(F - e) = r$
  - Simonovits'68: $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_{r+1}),\ \ n \geq n_0$
  - Mubayi'10: Asymptotic for $m \leq \mathrm{ex}(n, F) + \varepsilon_F n$
  - P.-Yilma $\geq$'15: Asymptotic for $m \leq \mathrm{ex}(n, F) + o(n^2)$
- Bipartite F
  - Conjecture (Erdős-Simonovits'82, Sidorenko'93):
    - Random graphs are optimal
  - ..., Conlon-Fox-Sudakov'10, Li-Szegedy $\geq$'15, Kim-Lee-Lee $\geq$'15, ...
  - Forcing Conjecture (Li-Szegedy): F biparite non-tree $\Rightarrow$ extremal graphons are constants

Thank you!