An Effective Model of Facets Formation

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Technion

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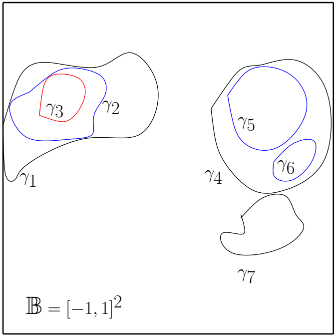
[1]Based on joint works with Senya Shlosman, Fabio Toninelli, Yvan Velenik and Vitali Wachtel

- A macroscopic variational problem.
- Low temperature 3D Ising model, Wulff shapes and (unknown) structure of microscopic facets.
- Facets on SOS surfaces with bulk Bernoulli fields.
- Fluctuations of level lines.

Surface tension τ_β and bulk susceptibility D_β are coming from an effective SOS-type model at inverse temperature β.



Nested family of loops $\mathcal{L} = (\gamma_1, \ldots, \gamma_7)$

- $\tau_\beta(\gamma) = \int_\gamma \tau_\beta(\mathfrak{n}_s)\mathrm{d}s$.
- $\tau_\beta(\mathcal{L}) = \sum_\ell \tau_\beta(\gamma_\ell)$.
- $\mathsf{a}(\gamma)$ - area inside γ.
- $\mathsf{a}(\mathcal{L}) = \sum_\ell \mathsf{a}(\gamma_\ell)$.

$$\min_{\mathcal{L}} \left\{ \frac{(\delta - \mathsf{a}(\mathcal{L}))^2}{2D_\beta} + \tau_\beta(\mathcal{L}) \right\} \qquad (\mathrm{VP}_\delta)$$

Let e be a lattice direction. Set

$$v = \frac{\delta}{\tau_\beta(\mathrm{e})\mathrm{D}_\beta},\ \ \sigma_\beta = \mathrm{D}_\beta\tau_\beta(\mathrm{e}) \text{ and } \tau(\cdot) = \frac{\tau_\beta(\cdot)}{\tau_\beta(\mathrm{e})}. \tag{1}$$

Since,

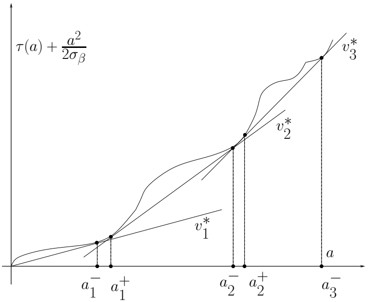
$$\frac{(\delta - \mathrm{a}(\mathcal{L}))^2}{2\mathrm{D}_\beta} + \tau_\beta(\mathcal{L}) = \frac{\delta^2}{2\mathrm{D}_\beta} + \tau_\beta(\mathrm{e})\left\{-v\mathrm{a}(\mathcal{L}) + \tau(\mathcal{L}) + \frac{\mathrm{a}(\mathcal{L})^2}{2\sigma_\beta}\right\},$$

we can reformulate the family of variational problems (VP_δ) as follows:

$$\min_{\mathcal{L}}\left\{-va(\mathcal{L}) + \tau(\mathcal{L}) + \frac{a(\mathcal{L})^2}{2\sigma_\beta}\right\}. \tag{VP_v}$$

$$\tau(a) = \min\{\tau(\mathcal{L}) \,:\, \mathrm{a}(\mathcal{L}) = a\}$$

Given $v \geq 0$, find $$\min_{\mathcal{L}} \left\{ -va + \tau(a) + \frac{a^2}{2\sigma_\beta} \right\}. \qquad (\mathrm{VP}_v)$$



If the graph of $a \mapsto \tau(a) + \frac{a^2}{2\sigma_\beta}$ is not convex, then an (infinite) sequence of first order transitions with:

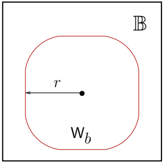
- Transition slopes $v_1^*, v_2^*, \dots$.
- Transition areas $a_\ell^\pm$.

Recall $\mathbb{B} = [-1,1]^2$. The rescaled surface tension $\tau(\mathrm{e}) = 1$. The Wulff shape

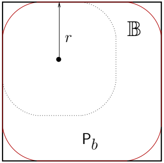
$$\mathrm{W} \triangleq \partial \left\{ x : x \cdot \mathfrak{n} \leq \tau(\mathfrak{n}) \ \forall \ \mathfrak{n} \in \mathbb{S}^1 \right\}$$

has radius 1. Consider:

$$\min_{\gamma \subset \mathbb{B};\, a(\gamma) = b} \tau(\gamma) \qquad (\mathrm{ST}_b)$$



Wulff Shape of area b



Wulff Plaquette of area b

Define $w = a(\mathrm{W})$.

- W_b solves (ST_b) for $b = r^2 w \in [0, w]$
- P_b solves (ST_b) for $b = 4 - r^2(4 - w) \in [w, 4]$.

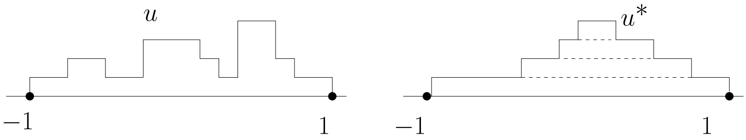
$$\min_{\mathcal{L}} \left\{ -va(\mathcal{L}) + \tau(\mathcal{L}) + \frac{a(\mathcal{L})^2}{2\sigma_\beta} \right\}. \qquad (\mathrm{VP}_v)$$

Define:

$$\mathsf{S}_b = \mathsf{W}_b \mathbb{1}_{b\in[0,w)} + \mathsf{P}_b \mathbb{1}_{b\in[w,4]}$$

Each nested family $\mathcal{L}$ of loops could be recorded as an integer valued function $u : \mathbb{B}_1 \mapsto \mathbb{N} \cup 0$. Set $b_\ell = |x : u(x) \geq \ell|$.
Rearrangement: $\mathcal{L}^* = \{\mathsf{S}_{b_1}, \mathsf{S}_{b_2}, \dots\}$ - nested family of loops.

$$\mathrm{a}(\mathcal{L}^*) = \mathrm{a}(\mathcal{L}) \text{ but } \tau(\mathcal{L}^*) \leq \tau(\mathcal{L}).$$



Hence only stacks of Wulff plaquettes and shapes matter.

Recall $w = a(\mathsf{W})$. For any $b \in (0, w)$, respectively, $b \in (w, 4)$,

$$\frac{\mathrm{d}}{\mathrm{d}b}\tau(\mathsf{W}_b) = \frac{1}{r(b)} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}b}\tau(\mathsf{P}_b) = \frac{1}{r(b)}.$$

Which means that optimal stacks of area a could be one of two types:

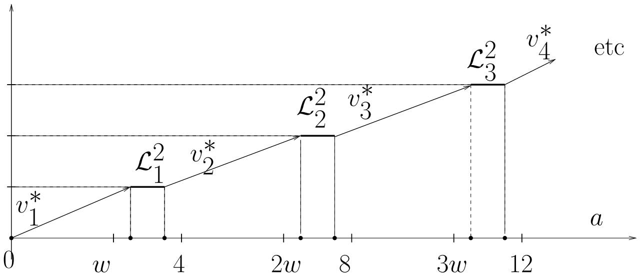
- Stacks $\mathcal{L}_\ell^1(a)$ of type 1. These contain $\ell - 1$ identical Wulff plaquettes and a Wulff shape, all of the same radius $r \in [0, 1]$.
- Stacks $\mathcal{L}_\ell^2(a)$ of type 2. These contain ℓ identical Wulff plaquettes of the same radius $r \in [0, 1]$.

Set $\ell^* \triangleq \frac{4}{4-w}$ (and assume $\ell^* \notin \mathbb{N}$). Then, relevant area ranges are:

$$Range(\mathcal{L}_\ell^1) = \begin{cases} [4(\ell-1), \ell w], & \text{if } \ell < \ell^* \\ [\ell w, 4(\ell-1)], & \text{if } \ell > \ell^* \end{cases} \quad \text{and} \quad Range(\mathcal{L}_\ell^2) = [\ell w, 4\ell]$$

$$\min_{\mathcal{L}} \left\{ -va(\mathcal{L}) + \tau(\mathcal{L}) + \frac{a(\mathcal{L})^2}{2\sigma_\beta} \right\}. \qquad (\mathrm{VP}_v)$$

• If $w \leq 2\sigma_\beta$, then stacks of type 1 are never optimal, and (recall $Range(\mathcal{L}_\ell^2) = [\ell w, 4\ell]$):

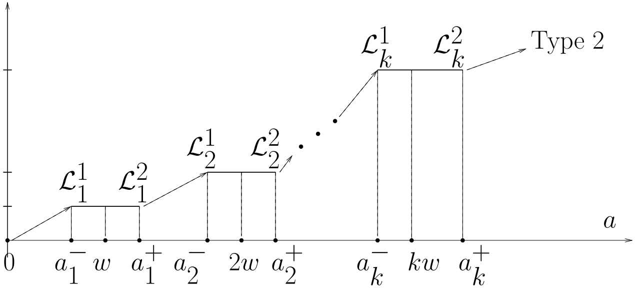


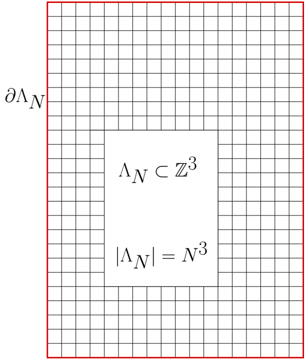
Transition slopes $0 < v_1^* < v_2^* < \dots$

Recall $\ell^* \triangleq \frac{4}{4-w} \notin \mathbb{N}$. For $\ell < \ell^*$ the area ranges are:

$$\mathsf{Range}(\mathcal{L}_\ell^1) = [4(\ell-1), \ell w] \ \text{ and } \ \mathsf{Range}(\mathcal{L}_\ell^2) = [\ell w, 4\ell]$$

- If $w > 2\sigma_\beta$, then then there exists a number $1 \leq k < \ell^* \left(1 - \frac{\sigma_\beta}{8}\right)$ such that stacks $\mathcal{L}_\ell^1$ show up for any $\ell = 1, \ldots, k$:





The Gibbs State

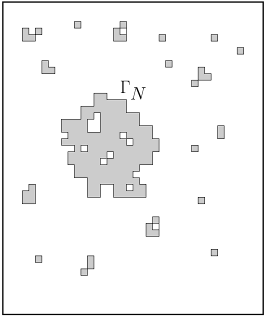
$$-\mathcal{H}_N^{-} = \frac{1}{2}\sum_{x\sim y}\sigma_x\sigma_y - \sum_{x\in\partial\Lambda_N}\sigma_x$$

$$\mathbb{P}_{N,\beta}^{-}(\sigma) \sim \mathrm{e}^{-\beta\mathcal{H}_N^{-}}$$

Low Temperature $\beta \gg 1 \ \Rightarrow m^*(\beta) > 0$.
Phase Segregation: Fix $m > -m^*$ and consider

$$\mathbb{P}_{N,\beta}^{m,-}(\cdot) = \mathbb{P}_{N,\beta}^{-}\left(\cdot \mid \sum \sigma_x = mN^3\right).$$

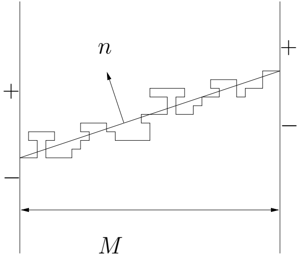
Typical Picture under $\mathbb{P}^{m,-}_{N,\beta}$



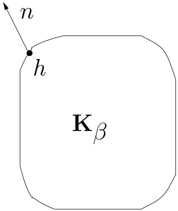
Volume of the microscopic Wulff droplet

$$|\Gamma_N| \approx \frac{m + m^*}{2m^*} N^3$$

Theorem (Bodineau, Cerf-Pisztora): As $N \to \infty$ the scaled shape $\frac{1}{N}\Gamma_N$ converges to the *macroscopic* Wulff shape.



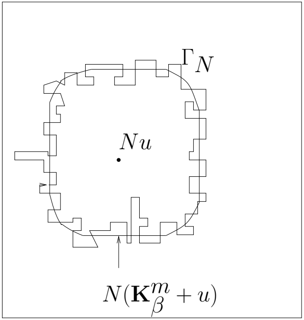
$$\xi_\beta(n) = -\lim_{M\to\infty} \frac{|\cos n|}{M^2} \log \frac{Z_M^{\pm}}{Z_M^{-}}.$$



$$\xi_\beta = \max_{h\in\partial\mathbf{K}_\beta} h\cdot n$$

Dilated Wulff Shape

$$\mathbf{K}_\beta^m = \left(\frac{m+m^*}{2m^*|\mathbf{K}_\beta|}\right)^{1/3} \mathbf{K}_\beta$$



Define (on unit box $\Lambda \subset \mathbb{R}^3$)

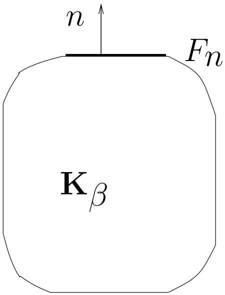
$$\phi_N(t) = \mathbb{1}_{\{Nt \in \Gamma_N\}} - \mathbb{1}_{\{Nt \notin \Gamma_N\}}.$$

Define

$$\chi^m(t) = \mathbb{1}_{\left\{t \in \mathbf{K}_\beta^m\right\}} - \mathbb{1}_{\left\{t \notin \mathbf{K}_\beta^m\right\}}$$

Then, under $\left\{\mathbb{P}_{N,\beta}^{m,-}\right\}$,

$$\lim_{N\to\infty} \min_u \|\phi_N(\cdot) - \chi^m(u + \cdot)\|_{\mathbb{L}_1(\Lambda)} = 0$$



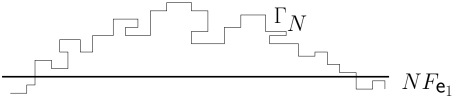
ξ_β - support function of $\mathbf{K}_\beta$. Then

$$F_n = \partial\xi_\beta(n).$$

Set e_i - lattice direction. Dobrushin '72, Miracle-Sole '94:

For $\beta \gg 1$ F_{e_i} is a proper 2D facet

Zooming Bodineau, Cerf-Pisztora picture, what happens?



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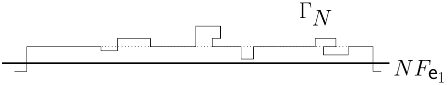

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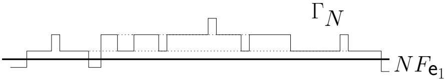
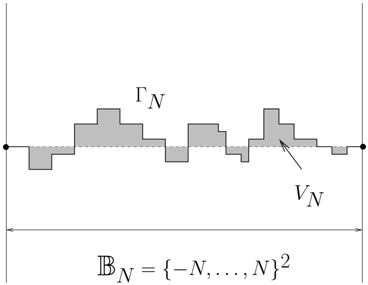
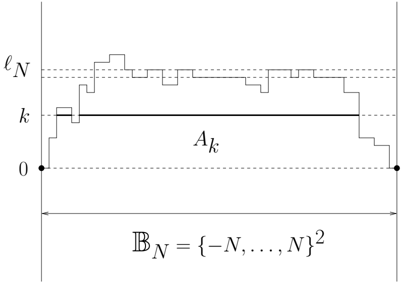






Bodineau, Schonmann, Shlosman '05

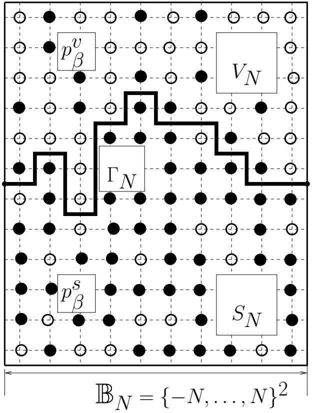
$$\mathbb{P}_N(\Gamma_N = \gamma) \sim e^{-\beta|\gamma|}$$

$$\mathbb{P}_N^m(\cdot) = \mathbb{P}_N\left(\cdot \middle| V_N \geq mN^3\right)$$

Result: There exists $a(\beta) \searrow 0$ such that

$$\ell_N = \max\left\{k : A_k \geq a(\beta)N^2\right\}$$

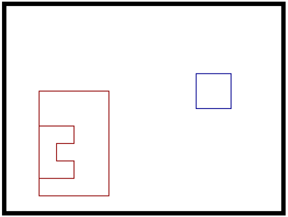
satisfies $A_{\ell_N - 1} \geq (1 - a(\beta))N^2$.

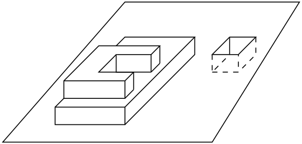


Configuration:
$\left(\Gamma_N, \{\xi^v_i\}_{i\in V_N}, \{\xi^s_j\}_{j\in S_N}\right)$.
Total number of particles:

$$\Xi_N = \sum_{i\in V_N} \xi^v_i + \sum_{j\in S_N} \xi^s_j$$

- $|\Gamma|$ - area of Γ
- $\mathbb{B}_p(\xi) = p^\xi(1-p)^{1-\xi}$
- β large

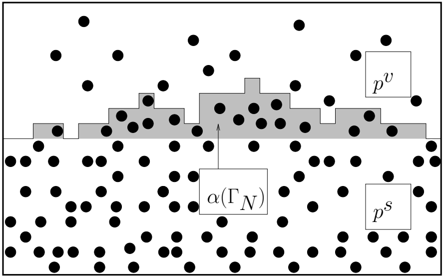




- Orientation of contours: Positive and negative (holes)
- $\alpha(\gamma)$ - signed area.
- $|\gamma|$ - length.
- Compatibility $\gamma \sim \gamma'$

For $\Gamma = \{\gamma_i\}$

$$|\Gamma| \sim \sum |\gamma_i|, \ \alpha(\Gamma) \triangleq \sum \alpha(\gamma_i)$$



Ξ_N - total number of particles

$$\mathbb{E}_N(\Xi_N) = \frac{p^s + p^v}{2} N^3 \triangleq pN^3$$

Consider

$$\mathbb{P}_N^A(\cdot) = \mathbb{P}_N\left(\cdot \middle| \Xi_N = pN^3 + AN^2\right)$$

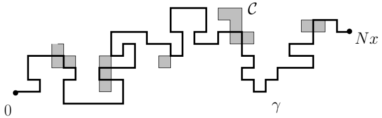
2D Surface Tension: $\log \mathbb{P}\left(\alpha(\Gamma_N) = aN^2\right) \approx -N\tau_\beta(a)$.

Bulk Fluctuations: $\Delta = 2(p^s - p^v)$, $\mathbb{E}_N\left(\Xi_N \middle| \alpha(\Gamma_N)\right) = pN^3 + \Delta\alpha(\Gamma_N)$.

$$\log \mathbb{P}_N\left(\Xi_N = pN^3 + AN^2 \middle| \alpha(\Gamma_N) = aN^2\right) \approx -\frac{(AN^2 - \Delta a N^2)^2}{N^3 R}$$

$$= -N\frac{(\delta - a)^2}{2D_\beta}, \quad \text{where } R = p^s(1-p^s) + p^v(1-p^v),$$

$D_\beta = R/(2\Delta^2)$ and $\delta = A/\Delta$. Hence $\min_a \left\{ \frac{(\delta-a)^2}{2D_\beta} + \tau_\beta(a) \right\}$.



$$w_\beta(\gamma) = \mathrm{e}^{-\beta|\gamma| - \sum_{\mathcal{C}\not\sim\gamma} \Phi_\beta(\gamma)}$$

$$G_\beta(Nx) = \sum_{\gamma:0\to Nx} w_\beta(\gamma).$$

$$\tau_\beta(x) = -\lim_{N\to\infty} \frac{1}{N} \log G_\beta(Nx).$$

$$\tau_\beta(\gamma) = \int_\gamma \tau_\beta(n_s)\mathrm{d}s.$$

Macroscopic Variational Problem

Recall $\Delta = 2(p^s - p^v)$, $R = p^s(1-p^s) + p^v(1-p^v)$, $D_\beta = R/(2\Delta^2)$ and $\delta = A/\Delta$.

$$(\mathrm{VP})_\delta \qquad \min_a \left\{ \frac{(\delta - a)^2}{2D_\beta} + \min_{a(\mathcal{L})=a} \tau_\beta(\mathcal{L}) \right\}.$$

Fix $\beta \gg 1$. Bulk fluctuations simplify analysis of $\mathbb{P}_N^A$. Recall the contour representation $\Gamma = \{\gamma_i\}$.

Lemma 1 (No intermediate contours). $\forall A > 0$ there exists $\epsilon = \epsilon(A) > 0$ such that

$$\mathbb{P}_N^A\left(\exists \gamma_i : \frac{1}{\epsilon}\log N \leq |\gamma_i| \leq \epsilon N\right) = o(1).$$

Lemma 2 (Irrelevance of small contours)

$$\mathbb{P}_N^A\left(\Big|\sum \alpha(\gamma_i)\mathbb{1}_{\{|\gamma_i| \leq \epsilon^{-1}\log N\}}\Big| \gg N\right) = o(1).$$

Definition: γ is large if $|\gamma| \geq \epsilon N$.

A. Fix $A > 0$ and forget about intermediate contours $\frac{1}{\epsilon}\log N \leq |\gamma| \leq \epsilon N$.
B. Expand with respect to small contours $|\gamma| \leq \frac{1}{\epsilon}\log N$.

For $\Gamma = \{\gamma_i\}$ collection of large contours the effective weight is

$$\hat{\mathbb{P}}_N(\Gamma) \propto \exp\left\{-\beta \sum |\gamma_i| - \sum_{\mathcal{C}\not\sim\Gamma} \Phi_\beta(\mathcal{C})\right\}.$$

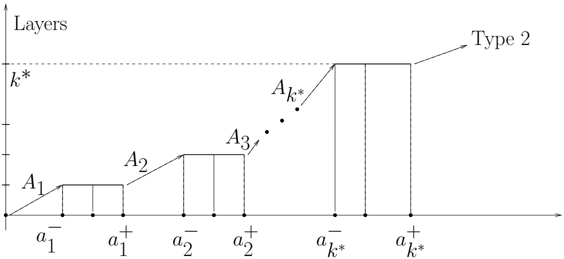
The family of clusters $\mathcal{C}$ depends on N and A. However cluster weights $\Phi_\beta(\mathcal{C})$ remain the same, and they are small: For all β sufficiently large

$$|\Phi_\beta(\gamma;\mathcal{C})| \leq c\mathrm{e}^{-\beta(\mathrm{diam}(\mathcal{C})+1)}$$

Reduced Model of Large Contours and Bulk Particles:

$$\hat{\mathbb{P}}_N(\Gamma, \xi^v, \xi^s) = \hat{\mathbb{P}}_N(\Gamma) \prod_{i\in\hat{V}_N} \mathbb{B}_{p_v}(\xi_i^v) \prod_{j\in\hat{S}_N} \mathbb{B}_{p_s}(\xi_j^s)$$

Recall: $\Delta = 2(p^s - p^v)$, $R = p^s(1-p^s) + p^v(1-p^v)$, $D_\beta = R/(2\Delta^2)$ and $\delta = A/\Delta$. $(\mathrm{VP})_\delta$ $\min_a \left\{ \frac{(\delta - a)^2}{D} + \min_{a(\mathcal{L})=a} \tau_\beta(\mathcal{L}) \right\}$.



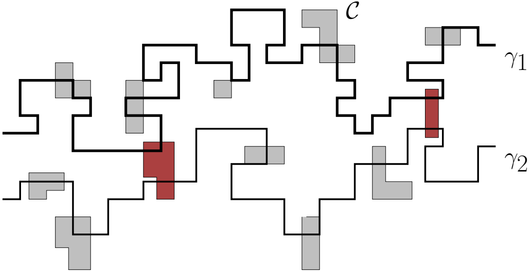
Set: $\hat{\mathbb{P}}_N^A(\cdot) = \hat{\mathbb{P}}_N\left(\cdot \middle| \Xi_N = pN^3 + AN^2\right)$. Then for any $\nu > 0$ and any $A \geq 0$, the (random) collection of large contours Γ satisfies:

$$\lim_{N\to\infty} \hat{\mathbb{P}}_N^A \left(\min_{\mathcal{L}^* - \text{solutions to } (\mathrm{VP})_\delta} \mathrm{d}_H\left(\frac{1}{N}\Gamma, \mathcal{L}^*\right) < \nu \right) = 1$$

1st Order (Shape) Transitions in Microscopic Models

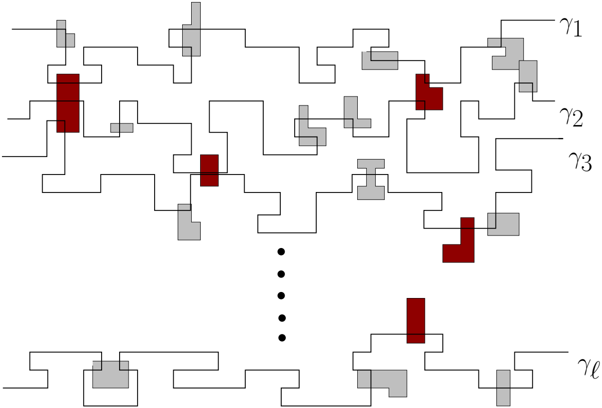
1. For $\beta \gg 1$ pure 2+1 SOS, conditioned to stay positive and with an additional bulk field $h > 0$, Chesi-Martinelli (JSP 1996) and Dinaburg-Mazel (JSP 1996) proved a sequence of layering transitions as $h \searrow 0$.
2. Spontaneous appearance of a droplet of linear size $N^{2/3}$ in the context of the 2D Ising model (any $\beta > \beta_c$) was established by Biskup, Chayes and Kotecky (CMP'03).
3. For $\beta \gg 1$ pure 2+1 SOS, conditioned to stay positive and with an additional attractive 0-layer boundary field h, Alexander, Dunlop and Miracle-Solé (JSP 2011) proved a sequence of layering transitions as $h \searrow 0$.
4. For $\beta \gg 1$ pure 2+1 SOS (without bulk Bernoulli fields) models of interfaces with zero b.c. on $\partial\mathbb{B}_N$, and conditioned to stay positive on $\mathbb{B}_N$, Caputo, Lubetzky, Martinelli, Sly and Toninelli proved in a series of works 2012-14 that there are $\lfloor \frac{1}{4\beta} \log N \rfloor$ macroscopic facets with asymptotically different Wulff Plaquette shapes.

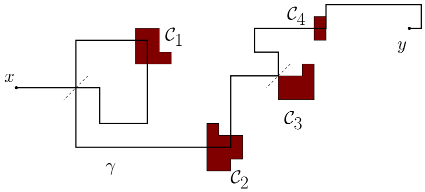
Difficulty: Control of Interactions of Contours in Macroscopic Stacks



$$\sum_{\mathcal{C} \not\sim \gamma_1 \cup \gamma_2} \Phi_\beta(\mathcal{C}) = \sum_{\mathcal{C} \not\sim \gamma_1} \Phi_\beta(\mathcal{C}) + \sum_{\mathcal{C} \not\sim \gamma_2} \Phi_\beta(\mathcal{C}) - \sum_{\mathcal{C} \not\sim \gamma_1 \cap \gamma_2} \Phi_\beta(\mathcal{C})$$

As $\beta \nearrow \infty$ the interaction becomes small, but fluctuations of contours (level lines) also become small, at least along axis directions. Hence we are dealing with small attraction vrs small entropic repulsion.


γ_1
γ_2
γ_3
γ_ℓ



$$w_\beta(\gamma) = \mathrm{e}^{-\beta|\gamma| + \sum_{\mathcal{C} \not\sim \gamma} \Phi_\beta(\gamma;\mathcal{C})}.$$

Assumption:

$$|\Phi_\beta(\gamma;\mathcal{C})| \leq c\mathrm{e}^{-\chi\beta(\mathrm{diam}(\mathcal{C})+1)}.$$

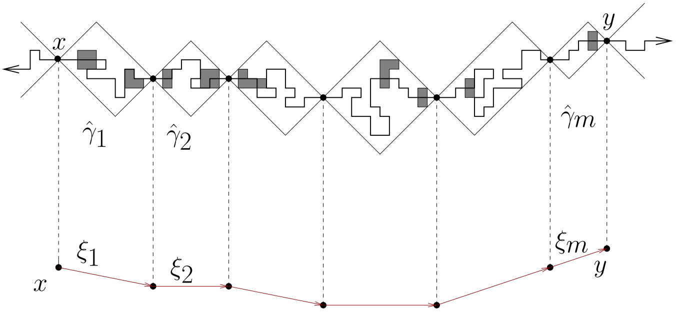
Theorem (I, Shlosman, Toninelli , JSP 2015)
If $\chi > \frac{1}{2}$, then repulsion wins over attraction for all β sufficiently large, in the sense that half-space surface tension equals to the full space surface tension.

Remarks:

a. In the case of SOS interfaces $\chi = 1$.

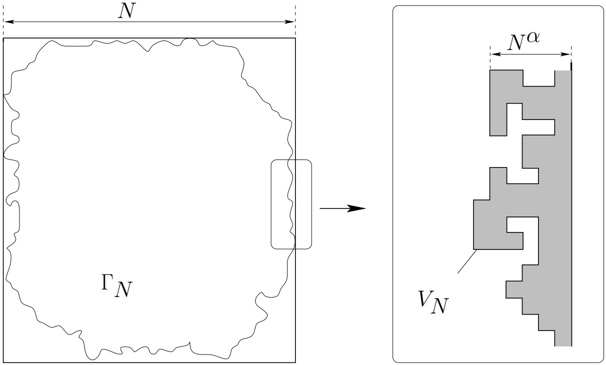
b. The theorem takes care of an interaction between one contour and a hard wall. Interactions between two, and more generally ℓ, ordered contours still has to be worked out.

Portion of a Contour Between x and y



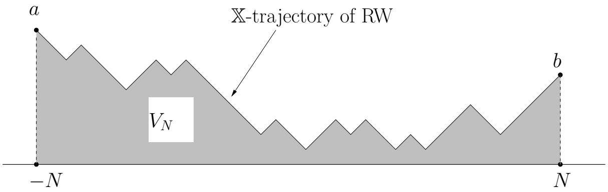
$$e^{\tau_\beta(y-x)} G_\beta(y-x) \cong \sum_m \sum_{\hat{\gamma}_1,\dots\hat{\gamma}_m} \prod \rho_\beta(\hat{\gamma}_i)$$

- $\{\rho_\beta(\cdot)\}$ is a probability distribution on the set of irreducible animals.
- $\xi_1 = (T_1, X_1), \xi_2 = (T_2, X_2), \dots$ steps of the effective random walk.



- Bulk fluctuation price for V_N is $\sim \frac{V_N N^2}{N^3} \sim \frac{V_N}{N}$.
- Repulsion price for staying N^α away from the boundary is $N^{1-2\alpha}$.

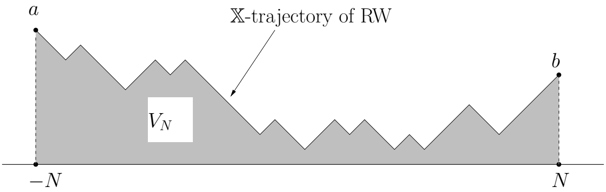
Therefore, $N^{1-2\alpha} \sim \frac{V_N}{N} \sim \frac{N^{1+\alpha}}{N} = N^\alpha$ gives $\alpha = 1/3$.



- Partition Function:

$$Z^{a,b}_{N,+,\lambda_N} = \sum_{\mathbb{X} \in \mathcal{T}^{a,b}_{N,+}} \mathrm{e}^{-\lambda_N V_N} \mathsf{p}(\mathbb{X})$$

- Scaling: $x_N(t) = \lambda_N^{1/3} \mathbb{X}(\lambda_N^{-2/3} t)$.



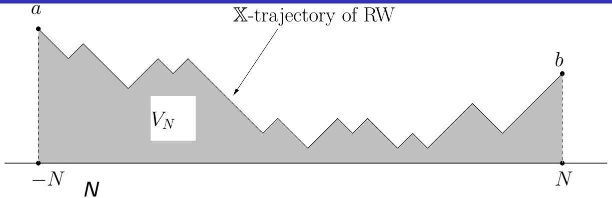
- Partition Function:

$$Z^{a,b}_{N,+,\lambda_N} = \sum_{\mathbb{X}\in\mathcal{T}^{a,b}_{N,+}} \mathrm{e}^{-\lambda_N V_N} \mathsf{p}(\mathbb{X})$$

- Scaling: $x_N(t) = \lambda_N^{1/3}\mathbb{X}(\lambda_N^{-2/3}t)$.

In general: $\{\Phi_\lambda\}$ - family of self-potentials, define $\Phi_\lambda(\mathbb{X}) = \sum_{-N}^{N}\Phi_\lambda(X_i)$.

$$Z^{a,b}_{N,+,\lambda_N} = \sum_{\mathbb{X}\in\mathcal{T}^{a,b}_{N,+}} \mathrm{e}^{-\Phi_{\lambda_N}(\mathbb{X})} \mathsf{p}(\mathbb{X})$$

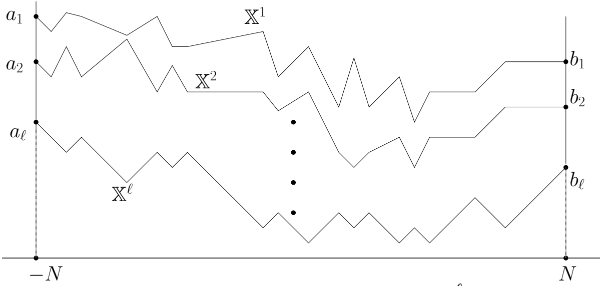


$$\Phi_\lambda(\mathbb{X}) = \sum_{-N}^{N} \Phi_\lambda(X_i) \text{ and } Z^{a,b}_{N,+,\lambda_N} = \sum_{\mathbb{X}\in\mathcal{T}^{a,b}_{N,+}} \mathrm{e}^{-\Phi_{\lambda_N}(\mathbb{X})}\mathsf{p}(\mathbb{X}).$$

Scale: $H_\lambda^2\Phi_\lambda(H_\lambda) = 1$.
Assumption: $\lim_{\lambda\to 0} H_\lambda^2\Phi_\lambda(H_\lambda r) = q(r)$ and $\lim_{r\to\infty} q(r) = \infty$.
Sturm-Liouville operator on $\mathbb{R}_+$: Set $\sigma^2 = \sum_x x^2\mathsf{p}(x)$ and $\mathcal{L} = \frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - q(r)$. Let φ_1 - the leading eigenfunction of $\mathcal{L}$.

Theorem (I, Shlosman, Velenik (CMP 2015)) The rescaled walk $x_N(t) = H^{-1}_{\lambda_N}\mathbb{X}(H^2_{\lambda_N} t)$ converges to ergodic diffusion with generator

$$\frac{\sigma^2}{2\varphi_1^2(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\varphi_1^2(r)\frac{\mathrm{d}}{\mathrm{d}r}\right).$$



$$Z^{\underline{a},\underline{b}}_{N,+,\lambda_N} = \sum_{\underline{\mathbb{X}} \in \mathcal{T}^{\underline{a},\underline{b}}_{N,+}} \mathrm{e}^{-\sum_{m=1}^{\ell} \Phi_{\lambda_N}(\mathbb{X}^m)} \prod_{m=1}^{\ell} \mathsf{p}(\mathbb{X}^m).$$

Work in Progress: Let $\varphi_1, \dots, \varphi_\ell$ be first ℓ eigenfunctions of $\mathcal{L}$. Define $\Delta_\ell(\underline{r}) = \det(\varphi_i(r_j))$. Then, the rescaled process $H^{-1}_{\lambda_N}\underline{\mathbb{X}}(H^2_{\lambda_N}t)$ converges to ergodic diffusion on $\{\underline{r} : 0 < r_\ell < \cdots < r_2 < r_1\}$ with generator

$$\frac{\sigma^2}{2\Delta^2_\ell(\underline{r})}\mathrm{div}\left(\Delta^2_\ell(\underline{r})\nabla\right).$$