Height fluctuations in interacting dimers

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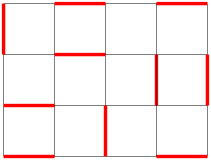
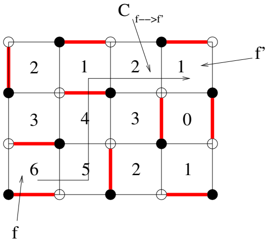
Limis shapes, ICERM, April 2015

Perfect matchings of $\mathbb{Z}^2$ and height function





Height function:

$$h(f') - h(f) = \sum_{b \in C_{f \to f'}} \sigma_b (1_{b \in M} - 1/4)$$

where $\sigma_b = +1/-1$ if b crossed with white on the right/left.

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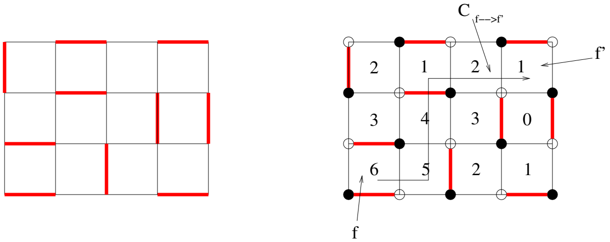

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Crucial observation: white-to-black flux $(1_{b \in M} - 1/4)$ is divergence-free. Important point: $\mathbb{Z}^2$ is bipartite.

Non-interacting dimers (or uniform perfect matchings)

If Λ is a large domain, e.g. the $2L \times 2L$ square/torus, many ($\approx \exp(cL^2)$) perfect matchings exist.

Call $\langle \cdot \rangle_{\Lambda;0}$ the uniform measure.

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Observe:

- By symmetry, on the torus, $\langle 1_{b \in M} \rangle_{\Lambda;0} = 1/4$ for every b, so that $\langle h(f) - h(f') \rangle_{\Lambda;0} = 0$.
- Dimers do not interact (except for hard-core constraint).

Non-interacting dimers (or uniform perfect matchings)

Known facts:

- Dimer-dimer correlations decay slowly:

$$\lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{b \in M}; 1_{b' \in M} \rangle_{\Lambda,0} \approx |b - b'|^{-2}$$

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- Height fluctuations grow logarithmically:

$$\lim_{\Lambda \to \mathbb{Z}^2} Var_{\Lambda,0}(h(f) - h(f')) \sim \frac{1}{\pi^2} \log |f - f'| \quad \text{as} \quad |f - f'| \to \infty$$

(see Kenyon-Okounkov-Sheffield for general bipartite graphs, periodic b.c.)

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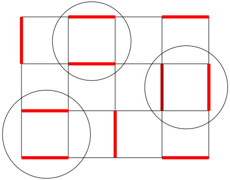
- the height field is asymptotically Gaussian: for $m\geq 3$, the m^{th} cumulant of $h(f)-h(f')$ is

$$\langle h(f)-h(f'); m\rangle_{\Lambda,0} = o(Var_{\Lambda,0}(h(f)-h(f'))^{m/2}).$$

  (recall: cumulants of X are zero for $m\geq 3$ iff X is Gaussian)

Interacting dimers

Associate an energy $\lambda \in \mathbb{R}$ to adjacent dimers:



I.e., with $N(M)$ the number of adjacent pairs of dimers in M,

$$\langle \cdot \rangle_{\Lambda,\lambda} = \frac{\sum_M e^{\lambda N(M)} \cdot}{Z_{\Lambda,\lambda}}$$

[Alet et al., Phys. Rev. Lett 2005]

Interacting dimers

Theorem [Giuliani, Mastropietro, T. 2014] If $|\lambda| \leq \lambda_0$ then:

- Fluctuations still grow logarithmically:

$$\lim_{\Lambda \to \mathbb{Z}^2} Var_{\Lambda,\lambda}(h(f) - h(f')) \sim \frac{K(\lambda)}{\pi^2} \log |f - f'|$$

  with $K(\cdot)$ analytic and $K(0) = 1$;

- for $m \geq 3$, the m^{th} cumulant of $h(f) - h(f')$ is *bounded*:

$$\sup_{f,f'} \lim_{\Lambda \to \mathbb{Z}^2} \langle h(f) - h(f'); m \rangle_{\Lambda,\lambda} \leq C(m).$$

Interacting dimers

- Convergence to the GFF

  If $|\lambda| \leq \lambda_0$ then convergence to Gaussian Free Field: if $\varphi \in C_0^\infty(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} \varphi(x)dx = 0$ then, as $\epsilon \to 0$,

$$\epsilon^2 \sum_f \varphi(\epsilon f) h(f) \Rightarrow \int_{\mathbb{R}^2} \varphi(x) X(x) dx$$

  with X the Gaussian Free Field of covariance

$$-\frac{K(\lambda)}{2\pi^2} \log|x-y|.$$

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- Theorem proven with periodic boundary conditions
- For $\lambda = 0$, Kenyon '00 proves conformal invariance of height moments e.g.

$$g_{\mathcal{D}}(x, y) = \lim_{L \to \infty} \langle (h_x - \langle h_x \rangle_{\Lambda,0})(h_y - \langle h_y \rangle_{\Lambda,0}) \rangle_{\Lambda,0}$$

(lattice spacing $1/L$ tends to zero, Λ is suitable discretization of domain $\mathcal{D} \subset \mathbb{C}$ and x, y tend to distinct points)

Challenge: proof for $\lambda \neq 0$

Non-interacting dimers: Kasteleyn theory

Partition functions and correlations given by determinants (or Pfaffians)

Define an antisymmetric $|\Lambda| \times |\Lambda|$ matrix K, indexed by lattice sites, as $K(x, x \pm e_1) = \pm 1, K(x, x \pm e_2) = \pm i$ and zero otherwise. Then,

$$Z = \sum_M 1 = Pf(K)$$

with, for antisymmetric $2n \times 2n$ matrix A,

$$Pf(A) = \frac{1}{2^n n!} \sum_\pi (-1)^\pi A_{\pi(1)\pi(2)} \cdots A_{\pi(2n-1)\pi(2n)}.$$

Non-interacting dimers: Kasteleyn theory

Similarly, if $b_1 = (x_1, x_2), b_2 = (x_3, x_4)$ are two bonds ($x_i \in \mathbb{Z}^2$, $|x_1 - x_2| = |x_3 - x_4| = 1$), then

$$\langle 1_{b_1 \in M} 1_{b_2 \in M} \rangle_{\Lambda,0} = K(b_1)K(b_2)Pf(M)$$

with M the 4×4 matrix with $M_{ij} = K^{-1}(x_i, x_j)$.

E.g.

$$\langle 1_{(x,x+e_1)\in M} 1_{(y,y+e_1)\in M} \rangle_{\Lambda,0}$$
$$= K^{-1}(x, x+e_1)K^{-1}(y, y+e_1) - K^{-1}(x, y+e_1)K^{-1}(y, x+e_1)$$

Inverse Kasteleyn matrix (or "propagator")

The inverse matrix K^{-1} can be computed explicitly, diagonalizing K:

$$\lim_{\Lambda\to\mathbb{Z}^2} K^{-1}(x,y) = \int_{[-\pi,\pi]^2} \frac{d\mathbf{k}}{(2\pi)^2} \frac{e^{-i\mathbf{k}(x-y)}}{-i\sin k_1 + \sin k_2}$$

Singularities at $(k_1,k_2) = (0,0),(\pi,0),(\pi,\pi),(0,\pi)$ produce $|x-y|^{-1}$ decay of K^{-1}:

$$K^{-1}(x,0) \overset{|x|\to\infty}{\sim} \frac{1}{2\pi}\left[\frac{1}{x_1+ix_2} + \frac{(-1)^{x_2}}{x_1-ix_2}\right]$$

Back to height fluctuations (free case)

Recall $h(f') - h(f) = \sum_{b \in C_{f \to f'}} \sigma_b (1_{b \in M} - 1/4)$
One finds

$$\sigma_b \sigma_{b'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{b \in M} ; 1_{b' \in M} \rangle_{\Lambda,0} = A_{b,b'} + B_{b,b'} + C_{b,b'}$$

$$= -\frac{1}{2\pi^2} \Re \left[\Delta z_b \Delta z_{b'} \frac{1}{(z_b - z_{b'})^2} \right]$$

$$+ Osc(z_b, z_{b'}) \frac{1}{|z_b - z_{b'}|^2} + O(|z_b - z_{b'}|^{-3}).$$

Then [Kenyon-Okounkov-Sheffield '06],

$$\sum_{b \in C_{f \to f'}, b' \in C'_{f \to f'}} A_{b,b'} \sim -\frac{1}{2\pi^2} \Re \int_f^{f'} \frac{dz dz'}{(z - z')^2} = \frac{1}{\pi^2} \log |f - f'|.$$

Dimer-dimer correlations, interacting case

Theorem If λ is small, then

$$\sigma_b \sigma_{b'} \lim_{\Lambda \to \mathbb{Z}^2} \langle 1_{b\in M}; 1_{b'\in M} \rangle_{\Lambda,\lambda}$$
$$= -\frac{K(\lambda)}{2\pi^2} \Re\Big[\Delta z_b \Delta z_{b'} \frac{1}{(z_b - z_{b'})^2}\Big]$$
$$+ Osc(z_b, z_{b'}) \frac{1}{|z_b - z_{b'}|^{2+\eta(\lambda)}} + O(|z_b - z_{b'}|^{-3+O(\lambda)}).$$

with $K(\cdot)$, $\eta(\cdot)$ analytic and $K(0) = 1, \eta(0) = 0$.

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with $K(\cdot)$, $\eta(\cdot)$ analytic and $K(0) = 1, \eta(0) = 0$.
Note:

- in the main term the critical exponent remains 2
- in the oscillating term it changes to $2 + \eta(\lambda)$ (non-universal).

Non-interacting dimers: "lattice free fermions"

Algebraic identity: Pfaffian can be written as "Grassmann Gaussian integral"

$\{\psi_x\}_{x\in\Lambda}$ Grassmann variables: $\psi_x\psi_y = -\psi_y\psi_x$ and $\psi_x^2 = 0$.

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Integration rules:

$$\int \prod_1^n d\psi_i \; \psi_n \dots \psi_1 = 1$$

and

$$\int \prod_1^n d\psi_i \; \psi_k \dots \psi_1 = 0 \qquad k < n.$$

Then,

$$Pf(K) = \int \prod_{u \in \Lambda} d\psi_u e^{-\frac{1}{2}(\psi, K\psi)}$$

and

$$K^{-1}(x,y) = \langle \psi_x \psi_y \rangle_0 := \frac{1}{Pf(K)} \int \prod_{u \in \Lambda} d\psi_u e^{-\frac{1}{2}(\psi, K\psi)} \psi_x \psi_y .$$

Also "fermionic Wick theorem":

$$\langle \psi_{x_1} \dots \psi_{x_{2n}} \rangle_0 = \sum_{\text{pairings } \pi} \sigma_\pi \langle \psi_{x_{\pi(1)}} \psi_{x_{\pi(2)}} \rangle_0 \times \cdots \times \langle \psi_{x_{\pi(2n-1)}} \psi_{x_{\pi(2n)}} \rangle_0$$

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"Fermions" because of anticommutation, "free" because exponential of quadratic form

Interacting dimers as interacting fermions

Similarly, the partition function of the interacting model is written as

$$Z_{\Lambda,\lambda} = \frac{1}{Pf(K)} \int \prod d\psi_x \exp\Big(-\frac{1}{2}(\psi, K\psi) + V(\psi)\Big) \equiv \Big\langle \exp(V(\psi)) \Big\rangle_{\Lambda,0}$$

with

$$V(\psi) = V_4(\psi) + V_6(\psi) + \dots,$$

and

$$V_4(\psi) = \lambda \sum_x \psi_x \psi_{x+e_1} \psi_{x+e_2} \psi_{x+e_1+e_2},$$

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NB: for finite Λ, these are just exact identities, V is a polynomial (finite degree).

Difficulties I: a combinatorial problem

Naif approach: perturbative expansion in λ

$$\Big\langle \exp(V(\psi)) \Big\rangle_{\Lambda,0} = \sum_n \frac{1}{n!} \langle V(\psi)^n \rangle_{\Lambda,0}.$$

Each expectation is computed via Wick's rule as sum of "Feynman diagrams". However, number of pairings is at least $(n!)^2$. Not summable.

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Solution: anticommutation rules $\Rightarrow$ relative signs $\Rightarrow$ gain a factor $n!$ (ideas form the '80s, QFT; e.g. Gawedzki-Kupiaienen '86,...).

Difficulties II: “infrared problem”

Due to slow decay of two-point function K^{-1}, many Fenynman diagrams are divergent (as $\Lambda \to \infty$).

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Constructive QFT (Benfatto, Brydges, Gallavotti, Gawedzki, Kupiainen, Rivasseau, Spencer...) provides the right tools to cure these problems.

Step 1: a change of variables

Decompose $K^{(-1)}(x,y) = \langle \psi_x \psi_y \rangle_0$ around the 4 singularities $p_1 = (0,0), p_2 = (\pi, 0), p_3 = (\pi,\pi), p_4 = (0,\pi)$:

$$K^{-1}(x,y) = \sum_{\gamma=1}^{4} \int \frac{d\mathbf{k}}{(2\pi)^2} \chi(\mathbf{k} - p_\gamma) \frac{e^{-i\mathbf{k}(x-y)}}{-i \sin k_1 + \sin k_2} \qquad (1)$$

i.e. rewrite

$$\psi_x = e^{ip_1 x}\psi_{x,1} + ie^{ip_2 x}\psi_{x,2} + ie^{ip_3 x}\psi_{x,3} + e^{ip_4 x}\psi_{x,4}$$

with

$$\langle \psi_{x,\gamma}\psi_{y,\gamma'}\rangle_0 = \delta_{\gamma,\gamma'} \int \frac{d\mathbf{k}}{(2\pi)^2} \chi(\mathbf{k}) \frac{e^{-i\mathbf{k}(x-y)}}{-i \sin k_1 + (-1)^{\gamma+1} \sin k_2}$$
$$\sim \frac{\delta_{\gamma,\gamma'}}{4\pi} \frac{1}{(x_1 - y_1) + i(-1)^{\gamma+1}(x_2 - y_2)}$$

Step 1: a change of variables

This way, $V(\psi)$ becomes

$$V(\psi) = \lambda \sum_{x} \psi_{x,1}\psi_{x,2}\psi_{x,3}\psi_{x,4} + \text{higher order},$$

Step 2: multi-scale integration

- multiscale decomposition of the "free propagator" or of the field:
$$\psi_{x,\gamma} = \psi^{(0)}_{x,\gamma} + \psi^{(-1)}_{x,\gamma} + \psi^{(-2)}_{x,\gamma} + \cdots :$$
for each $\psi^{(h)}_{x,\gamma}$, integration restricted to $\mathbf{k} \approx 2^h$;
- multiscale integration starting from short-distance scales: at each scale h, effective potential
$$V^{(h)}(\psi^{\le h}) = \lambda^{(h)} \sum_x \psi^{\le h}_{x,1}\psi^{\le h}_{x,2}\psi^{\le h}_{x,3}\psi^{\le h}_{x,4} + \text{higher order}$$
- flow equation for the effective coupling:
$\lambda^{(h)} = \lambda^{(h+1)} + \beta(\lambda^{(h+1)}, \ldots, \lambda^{(0)})$
- key question: behavior of $\lambda^{(h)}$ as $h \to -\infty$.

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Important fact: the function $\beta(...)$ vanishes asymptotically for $h \to -\infty$, and $\lambda^{(h)} \to \lambda_{-\infty} = \lambda + O(\lambda^2)$.

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Relies on works by Benfatto-Mastropietro on a related model (Thirring model) where, essentially, the denominator

$$-i \sin k_1 + (-1)^{j+1} \sin k_2$$

in $\langle \psi_{x,\gamma} \psi_{y,\gamma} \rangle_0$ is linearized and replaced by

$$-ik_1 + (-1)^{j+1} k_2.$$

Uniform smallness of $\lambda^{(h)}$ guarantees convergence of perturbation expansion.

Analogy with the 2D Ising model

Let $\mu_{\Lambda,0}$ be the Gibbs measure of the nearest-neighbor 2D Ising model at T_c, and $\mu_{\Lambda,\lambda}$ the one with Hamiltonian perturbed by $\lambda \sum_{x,y} v(x-y)\sigma_x\sigma_y$, at its critical point $T_c(\lambda)$.

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- The analog of dimer-dimer correlations are energy-energy correlations: if $|x-x'|=|y-y'|=1$

$$\mu_{\Lambda,0}(\sigma_x\sigma_{x'};\sigma_y\sigma_{y'}) \approx |x-y|^{-2}. \tag{2}$$

  Greenblatt-Giuliani-Mastropietro '12: if $|\lambda| \le \lambda_0$ and $v(\cdot)$ finite range, then (2) still true.

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- The analog of (square of) spin-spin correlations $\mu_{\Lambda,\lambda}(\sigma_x;\sigma_y)$ is the "electric correlator"

$$\mathcal{E}(f,f') = \langle e^{i\pi\alpha(h(f)-h(f'))}\rangle_{\Lambda,\lambda}, \qquad \alpha = 1.$$

Computation of $\mathcal{E}(f,f')$ is hard even for $\lambda = 0$, (Pinson '04, Dubedat '11).

Conclusions

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  - control of a non-local fermionic observable (height field) in a non-integrable case

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- Novelties:
  - match between constructive QFT methods (huge literature) and some (simple) discrete complex analysis ideas
  - control of a non-local fermionic observable (height field) in a non-integrable case
- While critical exponent of dimer-dimer correlations is not universal, large-scale GFF behavior is;
- To be done (major difficulties):
  - get rid of periodic b.c., work with general domains (necessary to study e.g. conformal invariance).
  - control the exponential of the height function. Our result suggests:

$$\mathcal{E}(f, f') \approx |f - f'|^{-\alpha^2 K(\lambda)/2}.$$