Generalized Jucys-Murphy Elements and Canonical Idempotents in Brauer Algebras

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joint work with:

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Plan of Talk / Motivation

1. *Canonical Idempotents in multiplicity-free families of algebras*
2. *Wedderburn–Artin Theorem for tower of Brauer algebras*
3. *Module Decomposition for Doty's Permutation modules*

Look for these boxes throughout.

Sage Math Wish List

For certain finite dimensional algebras:

- `some_alg(smaller_alg)`
- `some_alg.centralizer(elt_lst)`
- ...

Let's Study Irreducible Representations of $\mathfrak{S}_r$

- Character theory dictates: *equinumerous with the conj. classes in* $\mathfrak{S}_r$
- A simple calculation dictates: *equinumerous with partitions* ($\lambda \vdash r$)
- Where to look for λ?

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Idea #1: *Internally . . .* $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_r$

Setup:

- $\Bbbk$ - field (char. $p \geqslant 0$);
- V^0 - trivial rep. for $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$

Induce from the Young subgroup $\mathfrak{S}_\lambda \subseteq \mathfrak{S}_r$. *Hey, look, a lambda!*

$$M^\lambda \;:=\; \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_r}\left(V^0\right) \;=\; V^0 \otimes_{\Bbbk\mathfrak{S}_\lambda} \Bbbk\mathfrak{S}_r.$$

Let's Study Irreducible Representations of $\mathfrak{S}_r$

Idea #2: *Externally . . . weight space inside tensor space*

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Setup:

- $\Bbbk$ - field (char. $p \geqslant 0$);
- V - vec. space over $\Bbbk$ (dim. n, w. basis $\{e_j : 1 \leqslant j \leqslant n\}$)
- Act on $V^{\otimes r}$ by place permutation. *E.g.,* ($n = 4, r = 5$),

$$[e_3 \otimes e_4 \otimes e_3 \otimes e_1 \otimes e_2] * (1, 5, 2) = [e_4 \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_3].$$

Focus on simple tensors of weight λ. *E.g.,* $wt(e_3 e_4 e_3 e_1 e_2) = (1, 1, 2, 1)$.

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Focus on simple tensors of weight λ. *E.g.,* $wt(e_3 e_4 e_3 e_1 e_2) = (1, 1, 2, 1)$.

Hey, look, a lambda?

$$\tilde{M}^\lambda := \operatorname{span}\{e_J : J \in [n]^r; wt_i(J) = \lambda_i\}.$$

E.g., for $\lambda = (4,1)$, $\tilde{M}^\lambda = \langle e_{11112}, e_{11121}, e_{11211}, e_{12111}, e_{21111} \rangle$.

Let's Study Irreducible Representations of $\mathfrak{S}_r$

Happy Coincidence: $M^\lambda \simeq \tilde{M}^\lambda$.

UnHappy Fact: the M^λ are rarely irreducible (take char. $\Bbbk = 0$).
Look inside for the irreducible ("Specht") modules S^λ.

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Turning to Brauer algebras $\mathfrak{B}_n(z)$ . . .

- Hartmann–Paget ('06) use "Idea #1" to build permutation modules for $\mathfrak{B}_n(z)$.
  ▷ They find analogs of Specht and Young modules in this context.
- Doty ('12) uses "Idea #2" to build permutation modules for $\mathfrak{B}_n(z)$.
  ▷ We find Specht, and *perhaps* Young, modules in his context.

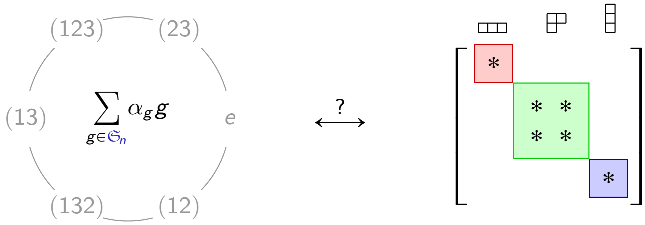
Interlude: Symmetric Group Algebras

- *The wrong way to find idempotents*
- *The right way to find idempotents*

The Symmetric Group Algebra $\mathbb{C}\mathfrak{S}_n$

- A semisimple algebra – simples indexed by partitions $\lambda \vdash n$
- Wedderburn–Artin decomp. – $\mathbb{C}\mathfrak{S}_n \cong \bigoplus_{\lambda \vdash n} M_{d_\lambda}(\mathbb{C})$

Example $(n = 3)$



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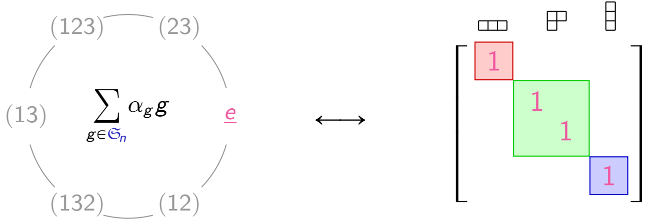
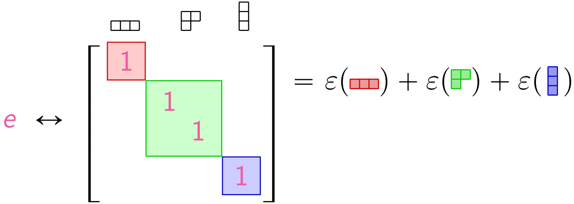


Notation & Goals

Find (nice) formulas for:

1. $\varepsilon(\lambda)$ – central idempotents (*identities for matrix blocks*). Unique.

Example ($\mathbb{C}\mathfrak{S}_3$)



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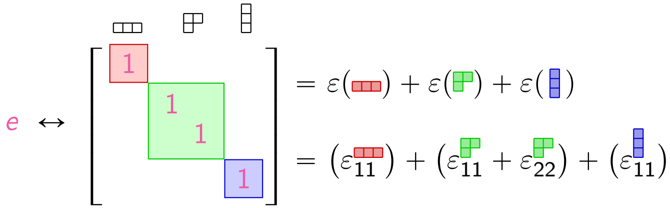

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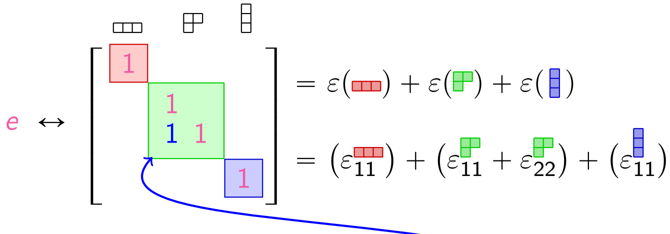


3. $\varepsilon^{\lambda}_{ij}$ ~~full set of d_λ^2 block matrix units, ex. ε_{21}~~ *not asking for these*

Theorem (Young, 1928)

1. *The central idempotents for $\mathbb{C}\mathfrak{S}_n$ are indexed by partitions of n.*
2. *The primitive idempotents for $\mathbb{C}\mathfrak{S}_n$ are indexed by standard Young tableaux of size n.*

Example ($\mathbb{C}\mathfrak{S}_n$)

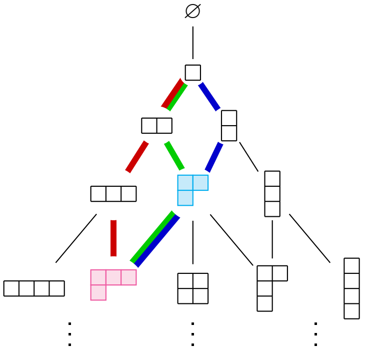
$$\varepsilon(\yng(3)) = e_{\young(123)} \qquad \varepsilon(\yng(2,1)) = e_{\young(12,3)} + e_{\young(13,2)} \qquad \varepsilon(\yng(1,1,1)) = e_{\young(1,2,3)}$$

Proof.

- $e_{\mathbf{T}}$ – defined via *row- (column-) (anti-)symmetrizers* $R_{\mathbf{T}}$ ($C_{\mathbf{T}}$).
- Proof Idea – study **intricate combinatorics** of interactions between $R_{\mathbf{T}}$ and $C_{\mathbf{S}}$ ... *15 pages(!) in Garsia's notes [Gar]* $\square$

Theorem (Vershik–Okounkov, 1996)

1. *Central idempotents for* $\mathbb{C}\mathfrak{S}_\bullet$ – *indexed by nodes in Young's lattice.*
2. *Primitive idempotents for* $\mathbb{C}\mathfrak{S}_\bullet$ – *indexed by paths in Young's lattice.*



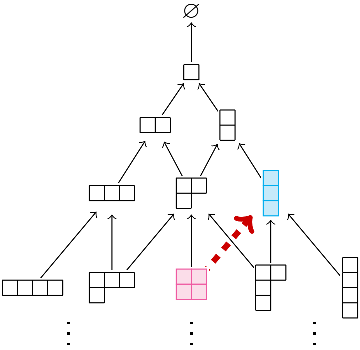
Example ($\mathbb{C}\mathfrak{S}_n$)

$$\varepsilon\left(\begin{smallmatrix}\square&\square\\\square\end{smallmatrix}\right) = \varepsilon_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} + \varepsilon_{\begin{smallmatrix}1&3\\2\end{smallmatrix}}$$

$$\varepsilon\left(\begin{smallmatrix}\square&\square&\square\\\square\end{smallmatrix}\right) = \varepsilon_{\begin{smallmatrix}1&2&3\\4\end{smallmatrix}} + \varepsilon_{\begin{smallmatrix}1&2&4\\3\end{smallmatrix}} + \varepsilon_{\begin{smallmatrix}1&3&4\\2\end{smallmatrix}}$$

Theorem (Vershik–Okounkov, 1996)

1. *Central idempotents for $\mathbb{C}\mathfrak{S}_\bullet$ – indexed by nodes in branching graph.*
2. *Primitive idempotents for $\mathbb{C}\mathfrak{S}_\bullet$ – indexed by paths in branching graph.*



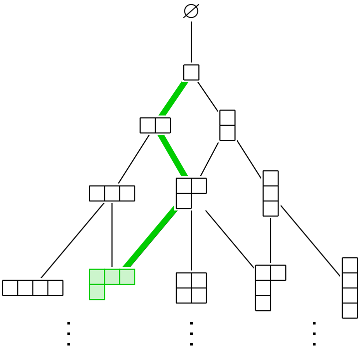
(Simple Restriction) Branching Graph

$$\mu \leftarrow \lambda \iff \mathrm{Hom}\big(S^\mu\ ,\ \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} S^\lambda\big) \neq 0$$

Wedderburn–Artin *Goals 1 & 2*

Theorem (Vershik–Okounkov, 1996; ...)

1. *Central idempotents for* $\mathbb{C}\mathfrak{S}_\bullet$ – *indexed by nodes in branching graph.*
2. *Primitive idempotents for* $\mathbb{C}\mathfrak{S}_\bullet$ – *= descending products of centrals.*



(Simple Restriction) Branching Graph

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- Ex. $\varepsilon_{\young(124,3)} := \varepsilon(\yng(3,1))\,\varepsilon(\yng(2,1))\,\varepsilon(\yng(2))\,\varepsilon(\yng(1))$

Proof. **Easy induction on** n. ✓ □

Sage Math Wish

```
sage: S3 = SymmetricGroupAlgebra(QQ, 3)
sage: S3.central_primitive_idempotent([2,1])
sage: S3.primitive_idempotent([[1,3], [2]])
```

Ditto for other (towers of) semisimple algebras.

End Interlude.

Schur–Weyl Duality

Schur '27:
Note that $\mathrm{GL}(V)$ *and* $\mathbb{C}\mathfrak{S}_n$ *acts on* $V^{\otimes n}$*:*

$$\mathrm{GL}(V) \circlearrowright V^{\otimes n} \circlearrowleft \mathbb{C}\mathfrak{S}_n$$

The two actions centralize each other:

- $\mathrm{End}_{\mathrm{GL}(V)} V^{\otimes n} \;=\; \mathbb{C}\mathfrak{S}_n$
- $\mathrm{End}_{\mathbb{C}\mathfrak{S}_n} V^{\otimes n} \;=\; \mathrm{span}_{\mathbb{C}}\, \mathrm{GL}(V)$

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Brauer '37:

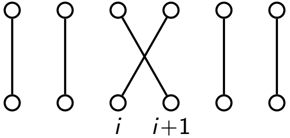
Now restrict to orthogonal matrices:

$$\begin{array}{ccccc} \mathrm{GL}(V) & \circlearrowright & V^{\otimes n} & \circlearrowleft & \mathbb{C}\mathfrak{S}_n \\ \cup\!\mid & & & & \mid\!\cap \\ \mathrm{O}(V) & & & & ?? \end{array}$$

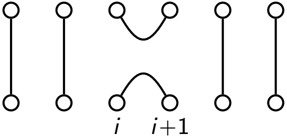
What is the corresponding centralizing object?

(It should be bigger than $\mathbb{C}\mathfrak{S}_n$*.)*

Generators:

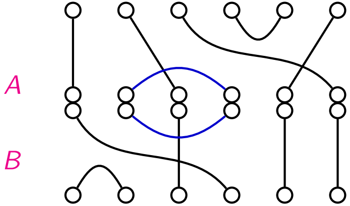


transpositions s_i $(1 \leqslant i < r)$

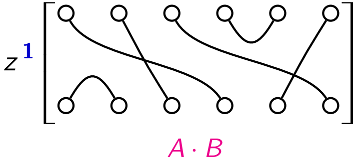


contractions c_i $(1 \leqslant i < r)$

Multiplication rule:



$=$



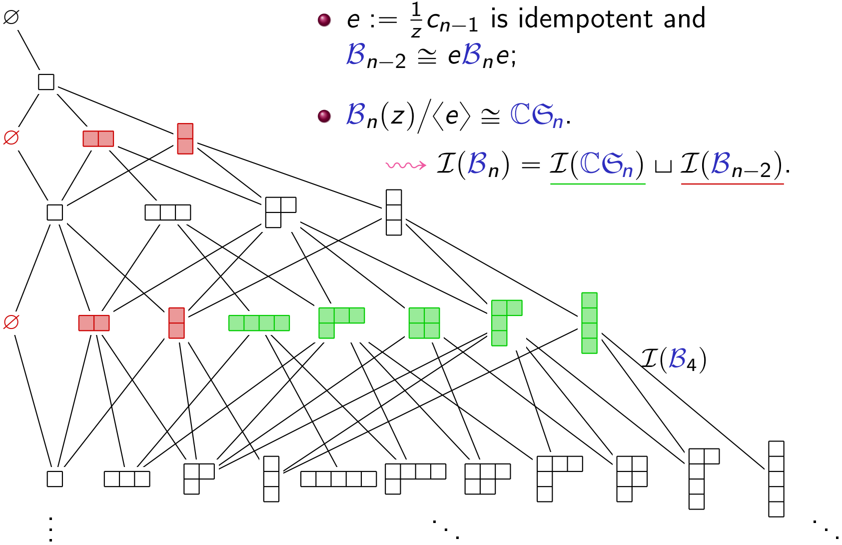
compose diagrams, top-to-bottom

exponent of z counts omitted internal loops

Irreducible Modules of $\mathcal{B}_n(z)$

- $e := \frac{1}{z} c_{n-1}$ is idempotent and $\mathcal{B}_{n-2} \cong e\mathcal{B}_n e$;
- $\mathcal{B}_n(z)/\langle e \rangle \cong \mathbb{C}\mathfrak{S}_n$.

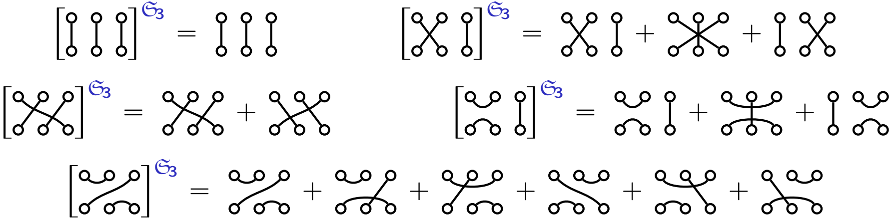
  $\rightsquigarrow \mathcal{I}(\mathcal{B}_n) = \mathcal{I}(\mathbb{C}\mathfrak{S}_n) \sqcup \mathcal{I}(\mathcal{B}_{n-2})$.



A Central Problem

- We'll look for central idempotents, indexed by $\lambda \vdash (n - 2\ell)$.
- It would be nice to have a natural basis of the center to get started.

Problem: Name $|\mathcal{I}(\mathcal{B}_3)|$=4 central linear combos of these $\mathfrak{S}_3$ orbit sums.



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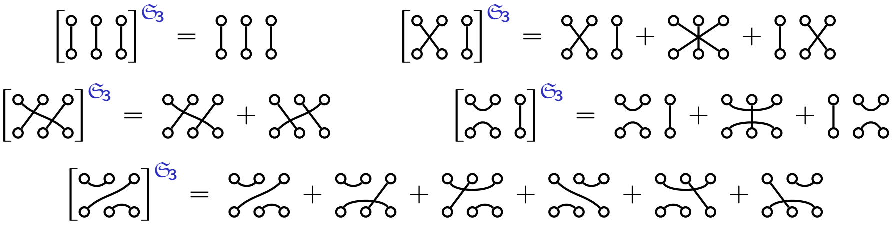

Sage Math Wish

In fact, any basis of the center will do (ask me why).

```
sage: BrauerAlgebra(3, z, F).center_basis()
```

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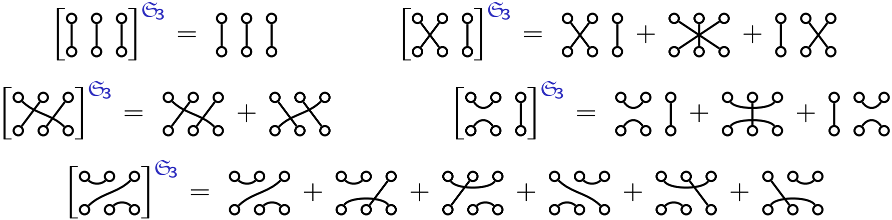


Sage Math Wish

In fact, any basis of the center will do (*Schur's Lemma*).

```
sage: BrauerAlgebra(3, z, F).center_basis()
```

Multiplicity Free Families & Jucys–Murphy Elements

- *Extension of [VO] to Multiplicity Free Families*
- *Utility of Jucys–Murphy elements for primitive/central idempotents*

Axiomatic Setup: MFFs

$\{\mathcal{A}_n : n \geqslant 0\}$ is a multiplicity-free family of algebras over $\mathbb{C}$ if:

- Each $\mathcal{A}_n$ is semisimple; with $\mathcal{A}_0 \cong \mathbb{C}$
- There are (unity-preserving) inclusions $\mathcal{A}_{n-1} \hookrightarrow \mathcal{A}_n$
- The multiplicity of $[\mu]$ in $\mathrm{Res}^{\mathcal{A}_n}_{\mathcal{A}_{n-1}}[\lambda]$ is **0** or **1**, $\forall \mu \in \mathcal{I}(\mathcal{A}_{n-1})$

Criterion

Restriction to $\mathcal{A}_{n-1}$ is multiplicity-free if and only if the centralizer algebra

$$Z(\mathcal{A}_{n-1}, \mathcal{A}_n) := \left\{x \in \mathcal{A}_n \mid xy = yx, \ \forall\, y \in \mathcal{A}_{n-1}\right\}$$

is commutative.

Examples. Alternating group algebras, Symmetric group algebras, Hecke algebras of types ABD, (affine & cyclotomic) Hecke–Clifford (super)algebras, BMW algebras, . . . , diagram algebras [GG], including **Brauer algebras** and Partition algebras.

Sage Math Wish

```
sage: S3 = SomeAlgebra(QQ, 3); S2 = SmallerAlgebra(QQ, 2)
sage: S3(S2.an_element())
sage: S3.centralizer(S2)
```

Main Results: MFFs

Theorem (DLS,'16)

Given an MFF,

1. *central idempotents* $\varepsilon(\lambda)$.
   - *may be computed as polynomials in* *Jucys–Murphy elements* *using Lagrange interpolation (see next slides).*
2. *primitive idempotents* $\varepsilon^{\lambda}_{ii} = \varepsilon_{\mathbf{T}}$.
   - *a complete system is given by taking products of descending central idempotents, i.e., nodes along the paths* $\mathbf{T}$.

Remark. The system is *canonical* in the sense that:

(1) no choices are made (aside from the embeddings $\mathcal{A}_{n-1} \hookrightarrow \mathcal{A}_n$);

(2) if any other system satisfies $e_{\mathring{\mathbf{T}}} e_{\mathbf{T}} = e_{\mathbf{T}}$ $(\forall \mathbf{T})$, then $e_{\mathbf{T}} = \varepsilon_{\mathbf{T}}$ $(\forall \mathbf{T})$.

Axiomatic Setup: JM Sequences

A sequence $(J_n \in \mathcal{A}_n : n \geqslant 1)$ is a (generalized) Jucys–Murphy sequence if $(\forall n)$:

- partial sums $J_1 + \cdots + J_{n-1} + J_n$ belong to the center $Z(\mathcal{A}_n)$;
- $\big\langle J_1, J_2, \ldots, J_n \big\rangle = \big\langle Z(\mathcal{A}_1), \ldots, Z(\mathcal{A}_{n-1}), Z(\mathcal{A}_n) \big\rangle = \mathrm{span}_{\mathbb{C}}\big\{\varepsilon_{\mathbf{T}} : |\mathbf{T}| = n\big\}$.

Proposition (DLS,'16)

JM sequences always exist for MFFs.

Computing the Coefficient Matrix $c_{\mathbf{T}}(k)$

- Write $J_k := \sum_{\mathbf{T}} c_{\mathbf{T}}(k)\varepsilon_{\mathbf{T}}$ $(\forall 1 < k \leqslant n)$. We wish to find the $c_{\mathbf{T}}(k)$'s.
- **Fact:** For any simple V of type λ, $\big(J_1 + \cdots + J_{n-1} + J_n\big)$ acts as a scalar a_λ on V.
- Given a path $\mathbf{T}$ in branching graph, let $\mathbf{typ}(\mathbf{T})$ denote terminal node, and let $\mathring{\mathbf{T}}$ denote the path $\mathbf{T} \setminus \mathbf{typ}(\mathbf{T})$.

Proposition (DLS,'16)

For all paths $\mathbf{T}$ *of length* n, *we have:*

$$c_{\mathbf{T}}(k) = c_{\mathring{\mathbf{T}}}(k) \text{ for all } k < n$$

$$c_{\mathbf{T}}(n) = a_{\mathbf{typ}(\mathbf{T})} - a_{\mathbf{typ}(\mathring{\mathbf{T}})}.$$

easy to compute

"Inverting" the Coefficient Matrix $c_{\mathbf{T}}(k)$

- Recall $J_k := \sum_{\mathbf{T}} c_{\mathbf{T}}(k)\varepsilon_{\mathbf{T}}$ for all $1 \leqslant k \leqslant n$.
- Given a path $\mathbf{T}$ of length n, define the interpolating polynomial

$$P_{\mathbf{T}}(x) := \prod_{\substack{|\mathbf{S}|=n \\ \mathbf{S}\neq\mathbf{T},\ \mathring{\mathbf{S}}=\mathring{\mathbf{T}}}} \frac{x - c_{\mathbf{S}}(n)}{c_{\mathbf{T}}(n) - c_{\mathbf{S}}(n)}$$

Theorem (DLS,'16)

The canonical idempotents are also given by the recursive formula

$$\varepsilon_{\mathbf{T}} = P_{\mathbf{T}}(J_n) \cdot \varepsilon_{\mathring{\mathbf{T}}}.$$

This finishes Goal 2.

Finding the Central Idempotents

- *exhaustive eigenvector search; or*
- *Kilmoyer's (generalized) Frobenius character formula; or*
- recursively compute using the interpolating polynomials . . .

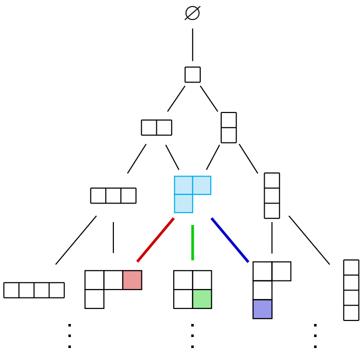
Theorem (DLS,'16)

- $P_{\mathbf{T}}(x)$ *depends only on* $\mu = \mathbf{typ}(\mathring{\mathbf{T}})$ *and* $\lambda = \mathbf{typ}(\mathbf{T})$*. Put* $P^{\lambda}_{\mu} := P_{\mathbf{T}}$*.*
- *For* $|\lambda| = n$*,*

$$\varepsilon(\lambda) = \sum_{\substack{\mu: \\ \mu \leftarrow \lambda}} P^{\lambda}_{\mu}(J_n)\, \varepsilon(\mu).$$

This finishes Goal 1.

Combinatorics / Content Vectors $c_{\mathbf{T}}$

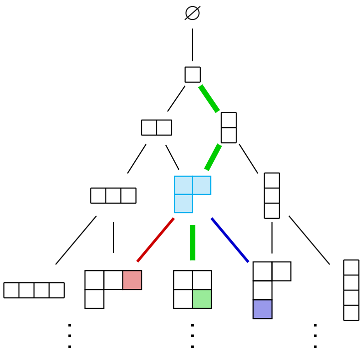


Example ($\mathbb{C}\mathfrak{S}_n$)

- Let (i, j) denote the coordinates of the last added box in **T**.
- Then $c_{\mathbf{T}}(n) = j - i$.

$$P(x) = \left(\frac{x - 2}{0 - 2}\right)\left(\frac{x - {}^{-}2}{0 - {}^{-}2}\right)$$

Combinatorics / Content Vectors $c_{\mathbf{T}}$



Example ($\mathbb{C}\mathfrak{S}_n$)

- Let (i, j) denote the coordinates of the last added box in $\mathbf{T}$.
- Then $c_{\mathbf{T}}(n) = j - i$.

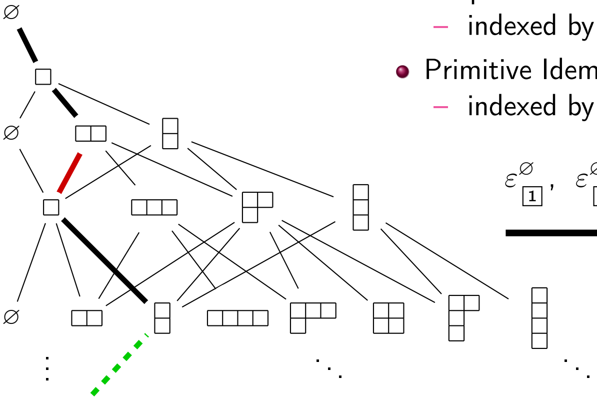
$$P^{(2,2)}_{(2,1)}(x) = \left(\frac{x - 2}{0 - 2}\right)\left(\frac{x - {}^{-}2}{0 - {}^{-}2}\right)$$

$$\varepsilon_{\begin{smallmatrix}1&3\\2&4\end{smallmatrix}} = P^{(2,2)}_{(2,1)}(J_4)\, P^{(2,1)}_{(1,1)}(J_3)\, P^{(1,1)}_{(1)}(J_2) = \left(\frac{J_4 - 2}{0 - 2}\right)\left(\frac{J_4 + 2}{0 + 2}\right) \cdot \left(\frac{J_3 + 2}{1 + 2}\right) \cdot \left(\frac{J_2 - 1}{-1 - 1}\right)$$

The Family $\mathcal{B}_\bullet(z)$ . . . *Forms an MFF*

Theorem (Wenzl, 1988)

$\mathcal{B}_n(z)$ is semisimple, with multiplicity-free restrictions, if $z \notin \mathbb{Z}$.



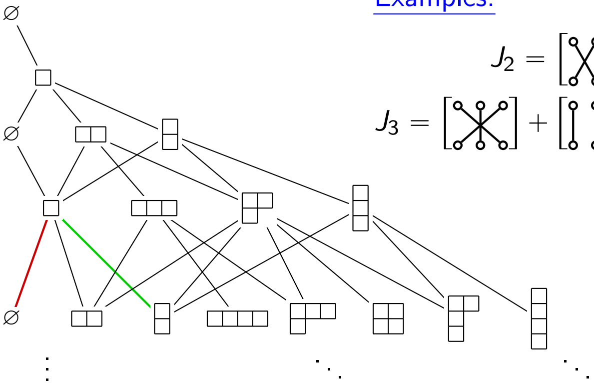
- Simples.
  - indexed by partitions $\lambda \vdash (n-2\ell)$.
- Primitive Idems.
  - indexed by up-down tableaux.

$\varepsilon^{\varnothing}_{\boxed{1}}$, $\varepsilon^{\varnothing}_{\boxed{1}\boxed{2}}$, $\varepsilon^{23}_{\boxed{1}}$, $\varepsilon^{23}_{\substack{\boxed{1}\\\boxed{4}}}$, $\varepsilon^{23,\,15}_{\boxed{4}}$

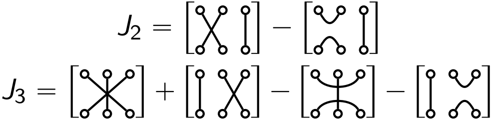
The Family $\mathcal{B}_\bullet(z)$ . . . *Has a nice JM-sequence*

Theorem (Nazarov, 1996; DLS,'16 (alternate proof))

The elements $J_k = \sum_{i<k} s_{ik} - \sum_{i<k} e_{ik}$ *form a JM-sequence.*



Examples.



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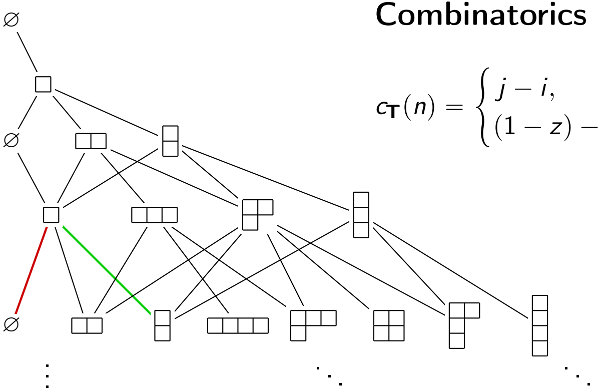
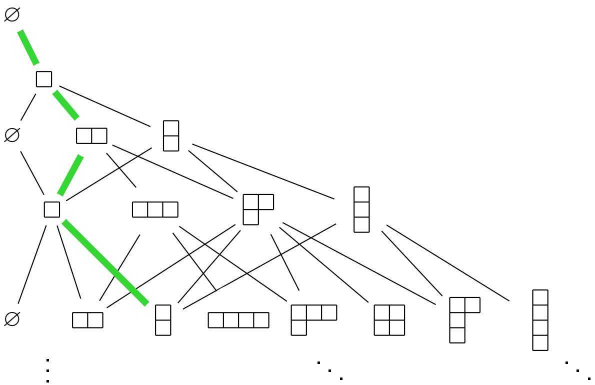


Combinatorics / Content Vectors

$$c_{\mathbf{T}}(n) = \begin{cases} j - i, & \text{if box added} \\ (1-z) - j + i, & \text{if box removed} \end{cases}$$



Example.

$$\varepsilon^{23}_{\begin{smallmatrix}1\\4\end{smallmatrix}} = \frac{(J_4 + z - 1)(J_4 + 1)}{2z} \cdot \frac{(J_3 + 2)(J_3 + 1)}{(z-1)(z-4)} \cdot \frac{(J_2 + z - 1)(J_2 - 1)}{2(2-z)}$$

Sage Math Wish

```
sage: B3 = BrauerAlgebra(3,z,F); B2 = BrauerAlgebra(2,z,F)
sage: B3(B2.an_element())
sage: B3.central_orthogonal_idempotents()
sage: B3.jucys_murphy(k)
```

Ditto for `PartitionAlgebra`, `AlternatingGroupAlgebra`, *and the like.*

Thanks!

[Gar] Garsia. Young's seminormal representation, Murphy elements, and content evaluations. unpublished, `lecture notes` (2003).

[GG] Goodman, Graber. On cellular algebras with Jucys Murphy elements. *J. Algebra* **330**, (2011).

[Naz] Nazarov. Young's orthogonal form for Brauer's centralizer algebra. *J. Algebra* **182** (1996), no. 3.

[VO] Vershik, Okounkov. A new approach to representation theory of symmetric groups. *Selecta Math.* **2** (1996), no. 4.

[Wen] Wenzl. On the structure of Brauer's centralizer algebras. *Ann. of Math. (2)* **128** (1988), no. 1.

arXiv:1606.08900

Extra slides

Using idempotents to study permutation modules

Central Idempotents Give Isotypic Components

- Consider permutation module for $\mathbb{C}\mathfrak{S}_3$ (*act by permuting coordinates*)

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = ??$$

$\mathbb{C}^3$

Central Idempotents Give Isotypic Components

- Consider permutation module for $\mathbb{C}\mathfrak{S}_3$ (*act by permuting coordinates*)
- Decompose into (irred.) Specht modules S^λ

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{S^{\square\square\square}} + \underbrace{\beta \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{??}$$

$$\mathbb{C}^3 = S^{\square\square\square} \oplus ??$$

Central Idempotents Give Isotypic Components

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$$\mathbb{C}^3 = S^{\square\square\square} \oplus ??$$

- What about the submodule $\left\{ \begin{bmatrix} \beta \\ \gamma - \beta \\ -\gamma \end{bmatrix} \right\}$? Is it $S^{\square\square / \square}$ or two one-dimensional modules?

To check, apply operators $\varepsilon(\square\square\square)$ *and* $\varepsilon\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right)$ *...*

$$\varepsilon(\square\square\square) * \begin{bmatrix} \beta \\ \gamma-\beta \\ -\gamma \end{bmatrix} = \left(\tfrac{1}{6}\textstyle\sum_g g\right) * \begin{bmatrix} \beta \\ \gamma-\beta \\ -\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \beta \\ \gamma-\beta \\ -\gamma \end{bmatrix} + \begin{bmatrix} \gamma-\beta \\ \beta \\ -\gamma \end{bmatrix} + \begin{bmatrix} \beta \\ -\gamma \\ \gamma-\beta \end{bmatrix} + \begin{bmatrix} \gamma-\beta \\ -\gamma \\ \beta \end{bmatrix} + \begin{bmatrix} -\gamma \\ \beta \\ \gamma-\beta \end{bmatrix} + \begin{bmatrix} -\gamma \\ \gamma-\beta \\ \beta \end{bmatrix} = \mathbf{0}$$

$$\varepsilon\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) * \begin{bmatrix} \beta \\ \gamma-\beta \\ -\gamma \end{bmatrix} = \left(\tfrac{1}{6}\textstyle\sum_g \mathrm{sign}(g)\, g\right) * \begin{bmatrix} \beta \\ \gamma-\beta \\ -\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \beta \\ \gamma-\beta \\ -\gamma \end{bmatrix} - \begin{bmatrix} \gamma-\beta \\ \beta \\ -\gamma \end{bmatrix} - \begin{bmatrix} \beta \\ -\gamma \\ \gamma-\beta \end{bmatrix} + \begin{bmatrix} \gamma-\beta \\ -\gamma \\ \beta \end{bmatrix} + \begin{bmatrix} -\gamma \\ \beta \\ \gamma-\beta \end{bmatrix} - \begin{bmatrix} -\gamma \\ \gamma-\beta \\ \beta \end{bmatrix} = \mathbf{0}$$

The Tensor Space Module for $\mathcal{B}_n(N)$

Setup:

- $V^{\otimes n}$ – basis is words in alphabet $[N]$ of length n.
- M^β – $\mathbb{C}\mathfrak{S}_n$-stable subspace, with basis $\{w \mid \text{multideg}(w) = \beta\}$
- Action of $\mathcal{B}_n(N)$ – depends on bilinear form defining $O(V)$;
  choose the following: $\langle e_i, e_j \rangle = \delta_{i,j'}$, where $j' := N + 1 - j$.
- Action on word $w = w_1 \cdots w_n$ – s_{ij} permutes places;
  $w * c_{12} = \delta_{w_1,(w_2)'} \sum_{a\in[N]} aa' w_3 \cdots w_n$.

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The M^β are not stable under $\mathcal{B}_n(N)$ action. Clump a few together...

(Doty, '12):

If $\mu \vdash (n{-}2\ell)$ has at most $N/2$ parts, then the $\mathcal{B}_n(N)$-stable subspace

$$D(\mu) \ := \ \bigoplus_{\alpha\in\Gamma(\ell,N/2)} M^{\mu+(\alpha\|\tilde{\alpha})},$$

where $\tilde{\cdot}$ is "reversal" and $\|$ is "concatenate," satisfies $V^{\otimes n} = \bigoplus_\mu D(\mu)$.

Finding Simples Inside the Permutation Modules $D(\mu)$

- Specht modules – Simples are $S(\mu) := S^\mu \otimes \mathcal{A}_\ell$ for $\mu \vdash (n-2\ell)$; S^μ is a Specht module for $\mathbb{C}\mathfrak{S}_{n-2\ell}$.
- $\mathcal{A}_\ell$ are the "half-diagram" modules with ℓ arcs.

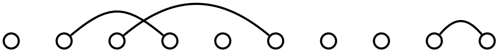
Theorem (DLS'18?)

- *The Specht module $S(\mu)$ is a submodule of $D(\mu)$, for $\mathcal{B}_n(\pm 2m)$ and for $\mathcal{B}_n(2m+1)$ for all char. $\Bbbk \neq 2$.*
- *$S(\mu)$ is part of a HUGE poset of submodules $C(\alpha)$ of $D(\mu)$ giving a filtration by the degenerate permutation modules $M^{\mu+(\alpha\|\tilde{\alpha})} \otimes \mathcal{A}_l$.*

Interlude (on Brauer Modules)

- *What does "$\mathcal{A}_\ell$" mean?*

$$\mathcal{D} = \{(2,4),(3,6),(9,10)\} \in \mathcal{A}_3$$



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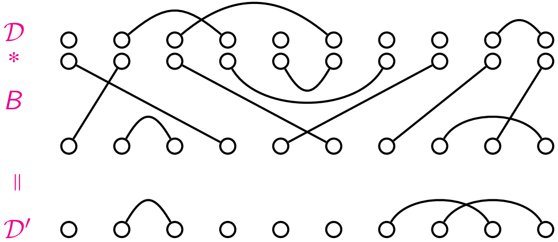

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- *What does $M \otimes \mathcal{A}_\ell$" mean?*

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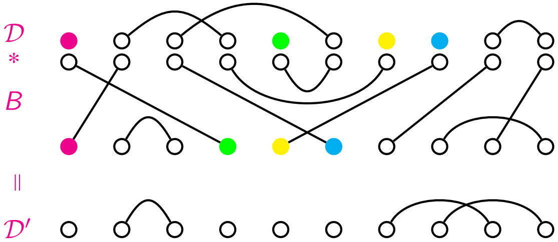
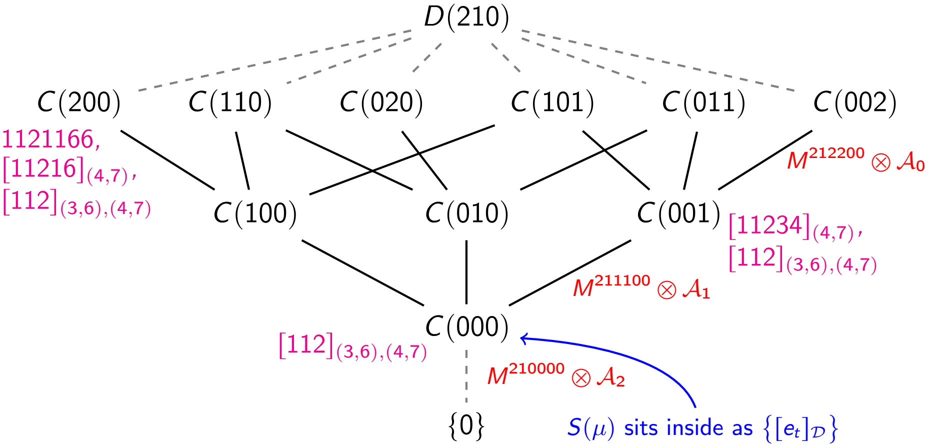


Let $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ act on M

Example Filtration $N = 6, n = 7, \mu = (2, 1)$

The poset of contraction submodules of $D(\mu)$



If $C(\beta) \gtrdot C(\alpha)$, then $C(\beta)/C(\alpha) \simeq M^{\mu+(\beta\|\tilde{\beta})} \otimes \mathcal{A}_I$ for some I.