TROPICAL GEOMETRY EXERCISES

ICERM NONLINEAR ALGEBRA BOOTCAMP

For these exercises you are encouraged to look at the help of the packages mentioned during the talk.

(1) Consider $\mathbb{Q}$ with the 3-adic valuation. Draw the tropicalizations of the curves $V(f) \subset \mathbb{P}^2$ for the following f:
  (a) $f = 3x + 6y - 5z$;
  (b) $f = 3x^2 + 4xy + 6y^2 + 7xz + 8yz + 9z^2$;
  (c) $f = 81x^3 + 9x^2y + 18xy^2 - 81y^3 + 3x^2z + xyz + 6y^2z + 4xz^2 + 7yz^2 + 9z^3$.

(2) Prove the tropical quadratic formula: The roots of $a \odot x^2 \oplus b \odot x \oplus c$ are

$$\begin{cases} b-a, c-b & \text{if } 2b \leq a+c \\ 1/2(c-a) & \text{if } 2b > a+c \end{cases}$$

What is the tropical cubic formula? Quartic? Quintic? Are you surprised by the fact that you can write down a tropical quintic formula?

(3) Consider the tropicalization of the cubic surface $V(t^{143}x_0^3 + x_0^2x_1 + t^{64}x_0^2x_2 + t^{122}x_0^2x_3 + x_0x_1^2 + t^2x_0x_1x_2 + x_0x_1x_3 + t^{15}x_0x_2 + t^{55}x_0x_2x_3 + t^{107}x_0x_3^2 + t^{36}x_1^3 + t^{23}x_1^2x_2 + t^{39}x_1^2x_3 + t^{16}x_1x_2^2 + t^{14}x_1x_2x_3 + t^{48}x_1x_3^2 + t^{12}x_2^3 + t^{12}x_2^2x_3 + t^{49}x_2x_3^2 + t^{95}x_3^3) \subseteq \mathbb{P}^3_{\mathbb{C}(t)}$. How many two-dimensional cells does it have? How many one-dimensional cells? How many vertices? Smooth cubic surfaces famously contain 27 lines; can you find the tropicalization of any of these?

(4) Let $I = \langle x_{ij} - x_{1i} + x_{1j} : 2 \leq i < j \leq n \rangle \subseteq \mathbb{C}[x_{12}^{\pm 1}, \ldots, x_{(n-1)n}^{\pm 1}]$ for $n \geq 5$. Compute $\operatorname{trop}(V(I))$ for the first few n. How many maximal cones does each have? Can you make/prove a conjecture?

(5) Consider the ideal $I = \langle 3x_1 + 2x_2 - 5x_3 + 4x_4, x_1 + x_2 - x_3 + 17x_4, 5x_1 + 10x_2 + 15x_3 - x_4 \rangle \subseteq \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}]$, where $\mathbb{C}$ has the trivial valuation. Is the tropicalization $\operatorname{trop}(V(I))$ equal to the intersection of the tropicalizations of the hypersurfaces given by the three generators of this ideal?

(6) Consider the ideal $I = \langle t^{101}x^2 + 37xy + 43x + t^{137}y^2 + 71y + t^{56}, t^{29}x^2 + 37t^{340}xy + 43t^{340}x + t^{725}y^2 + 17t^{340}y + t^{31}, t^2x + t^{140}y + 3\rangle \subset \mathbb{C}(t)[x, y]$. Is the variety $V(I) \subset \mathbb{A}^2$ empty?

(7) Consider the tropical hyperplane arrangement given by the hyperplanes $H_i = \text{trop}(V(l_i))$ for $1 \leq i \leq 4$, where $\mathbb{Q}$ has the 2-adic valuation:
   (a) $l_1 = 3x + 2y + 1$;
   (b) $l_2 = 5x + 7y + 2$;
   (c) $l_3 = x + 4y + 12$;
   (d) $l_4 = 8x - y + 4$.

   How many regions does this hyperplane have?