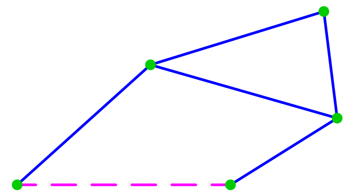
Problems for the ICERM NLA Bootcamp

Frank Sottile, 5 September 2018

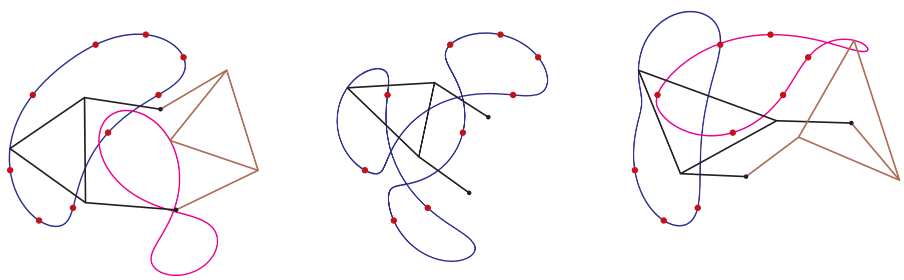
Alt's Problem. A *four-bar linkage* is a planar linkage consisting of a triangle with two of its vertices connected to two bars, whose other endpoints are fixed. The base of the triangle, the two attached bars, and the implied bar between the two fixed points are the four bars.



A general linkage has a one-dimensional constrained motion during which the joints may rotate, and the curve traced by the apex of the triangle is its *workspace curve*.

Alt's problem asks for the four-bar linkages whose workspace curve interpolates nine given points. Morgan, Sommese, and Wampler formulated and solved a polynomial system that models Alt's problem. They found that for nine complex points in general position, there are 1442 isolated solutions (occuring in triples of Robert's cognates). While the number 1442 has been certified, I do not know this number has been proven.

When the nine points are real, natural questions include how many of the 1442 are real (e.g. what are the possible numbers of real points). Even knowing that does not address the more subtle question of how many (if any) are mechanically desirable.

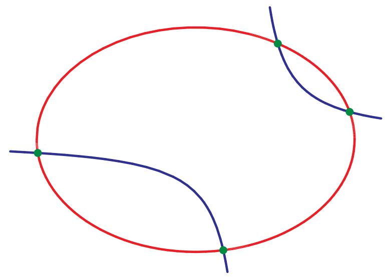


Here are some questions to chew on.

1. What are the possible numbers of real solutions to Alt's problem?
2. What are the possible numbers of mechanically desirable solutions?
3. Why is there one degree of freedom in a four-bar mechanism?
4. What is the workspace curve, as a curve?
5. What is its degree, as a plane curve?
6. Does the workspace curve have any attractive classical geometry?
7. As the number of real solutions to Alt's problem is too hard, find a simpler variant that you can solve.

Problems like this abound in nonlinear algebra.

Conics. A conic in the plane is given by six parameters $\{f_{ijk} \mid 0 \leq i,j,k \text{ and } i+j+k=2\}$ corresponding to its defining equation $f = \sum_{i,j,k} f_{ijk} x^i y^j z^k = 0$. Bézout's Theorem implies that if f and g are general, then they have four common zeroes, $\#\mathcal{V}(f,g) = 4$.



1. Can you give a proof of Bézout's Theorem in the plane? How about for two conics?
2. Can you give a proof of Bézout's Theorem in the plane to an Engineer?
3. Say as much as possible about $\{(f,g) \mid \#\mathcal{V}(f,g) \neq 4\}$.
4. Give a complete understanding of the set of real f,g with $\#\mathcal{V}(f,g)_{\mathbb{C}} = 4$ and

$$\mathcal{V}(f,g)_{\mathbb{R}} = \emptyset \; ? \qquad \#\mathcal{V}(f,g)_{\mathbb{R}} = 2 \; ? \qquad \#\mathcal{V}(f,g)_{\mathbb{R}} = 4 \; ?$$

There is a script `G28.sing` in the Dropbox that formulates a different Schubert problem.

Some nonlinear linear algebra. This is some Schubert calculus, which is a source of many interesting nonlinear problems that are a laboratory for many questions involving geometric problems.

Fix four 4-dimensional linear spaces $K_1, \ldots, K_4 \subset \mathbb{C}^8$ that are in general position. Determine the 4-dimensional linear spaces $H \subset \mathbb{C}^8$ with the following special position,

$$\dim H \cap K_i \;\geq 2 \quad \text{for } i = 1, \ldots, 4\,.$$

1. When each K_i is real, how many of these can be real ? (Do some experimentation.)
2. Formulate this as a system of polynomial equations in 16 variables.
3. Use some simple linear algebra to formulate this as a system of polynomial equations in 8 variables.
4. Use a bit more linear algebra to formulate this in terms of 2-dimensional invariant subspaces of an operator on $\mathbb{C}^4$.

Open-ended question about quadrics. Three quadrics in general position in $\mathbb{C}^3$ define eight points. Suppose you need find these points billions of times, equivalently, that this needs to be done really, *really* fast.

What is the best way to accomplish this task?

This may require testing different algorithms in different implementations, and also try different computational paradigms.