Nonlinear Algebra Bootcamp – SOS

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1. Consider the polynomial $p(x) = x^4 + 2ax^2 + b$. For what values of (a, b) is this polynomial nonnegative? Draw the region of nonnegativity in the (a, b) plane. Where does the discriminant of p vanish? How do you explain this?

2. Let $M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ be the Motzkin polynomial. Show that $M(x, y, z)$ is not SOS, but $(x^2 + y^2 + z^2)M(x, y, z)$ is.

3. Give a rational certificate of the nonnegativity of the trigonometric polynomial $p(\theta) = 5 - \sin\theta + \sin 2\theta - 3\cos 3\theta$.

4. Consider the polynomial system $\{x + y^3 = 2, x^2 + y^2 = 1\}$.
   (a) Is it feasible over $\mathbb{C}$? How many solutions are there?
   (b) Is it feasible over $\mathbb{R}$? If not, give a Positivstellensatz-based infeasibility certificate of this fact.

5. Consider the butterfly curve in $\mathbb{R}^2$, defined by the equation $x^6 + y^6 = x^2$. Give an SOS certificate that the real locus of this curve is contained in a disk of radius 5/4. Is this the best possible constant?

6. Consider the quartic form in four variables
$$p(w, x, y, z) := w^4 + x^2y^2 + x^2z^2 + y^2z^2 - 4wxyz.$$
   (a) Show that $p(w, x, y, z)$ is not a sum of squares
   (b) Find a multiplier $q(w, x, y, z)$ such that $q(w, x, y, z)p(w, x, y, z)$ is a sum of squares.

7. Let I be an ideal that is zero-dimensional and radical. Show that $p(x) \geq 0$ on $V(I)$ if and only if $p(x)$ is SOS mod I.

8. The *stability number* $\alpha(G)$ of a graph G is the cardinality of its largest stable set. Define the ideal $I = \langle x_i(1 - x_i) \text{ for } i \in V,\ x_ix_j \text{ for } ij \in E\rangle$.
   (a) Show that $\alpha(G)$ is *exactly* given by
$$\min \gamma \ \text{ such that } \ \gamma - \sum_{i \in V} x_i \text{ is SOS mod } I$$

(b) Recall that a polynomial is 1-SOS if it can be written as a sum of squares of affine (degree 1) polynomials. Show that an upper bound on $\alpha(G)$ can be obtained by solving

$$\min \gamma \quad \text{such that} \quad \gamma - \sum_{i \in V} x_i \text{ is 1-SOS mod } I$$

(c) Show that the given generators of the ideal I are already a Gröbner basis. Show that there is a natural bijection between standard monomials and stable sets of G.

(d) As a consequence of the previous fact, show that $\alpha(G)$ is equal to the degree of the Hilbert function of $\mathbb{R}[x]/I$.

Now let $G = (V, E)$ be the Petersen graph.

(e) Find a stable set in the Petersen graph of maximum cardinality

(f) Solve the relaxation from (8b) for the Petersen graph. What is the corresponding upper bound?

(g) Compute the Hilbert function of I, and verify that this answer is consistent with your previous results.