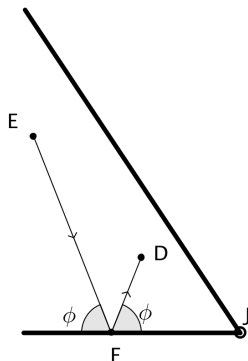


Billiards in Near Rectangle

Yang Yilong Chang Hai Bin

August 6, 2012

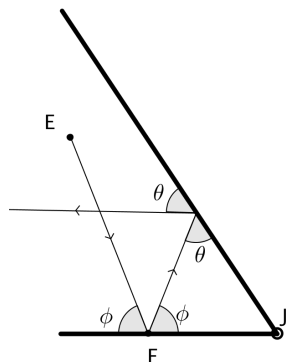
Introduction



Billiard ball starts at a point(E), with a given initial direction.

Whenever it hits a side, the trajectory will satisfy:
Angle of incidence
= angle of reflection

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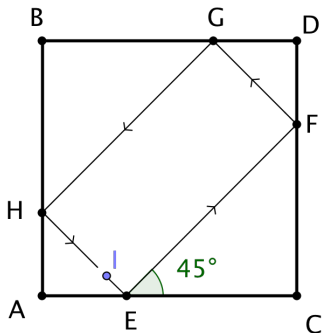
Whenever it hits a side, the trajectory will satisfy:

Angle of incidence
= angle of reflection

Ignore cases when: Billiard ball hits vertex (e.g. J)

Periodic Billiard Path:

if the ball comes back to the initial position with initial velocity direction.

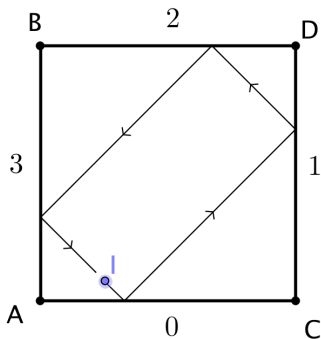


Word:

After labelling the sides of polygon using numbers/letters, record down the sequence of sides which the periodic path bounces off.

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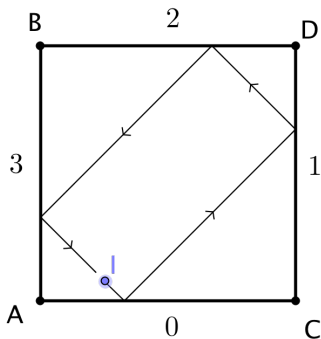
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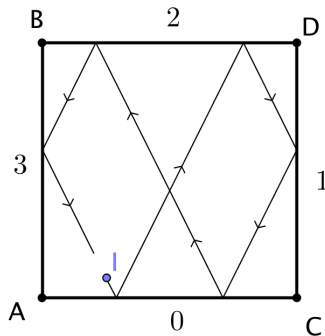
0123

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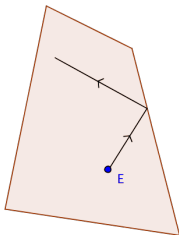
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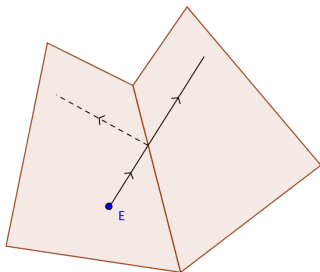
Unfolding:

Whenever the ball hits a side, reflect the polygon, and keep the “billiard path” straight. The path on the new polygon will “corresponds” to the trajectory in original polygon.



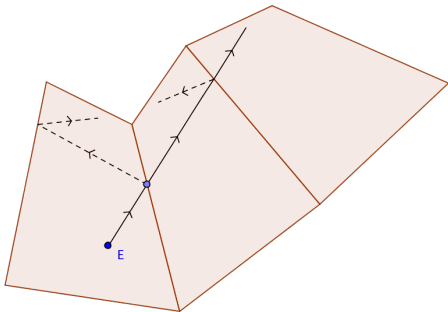
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there exists a periodic billiard path that hits the sides of P according to the order given by W .

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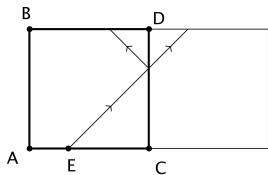
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Equivalent to saying that:

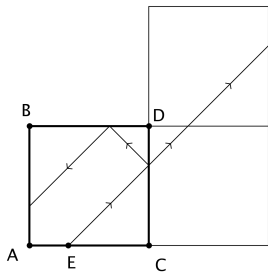
- 1 the first and last polygon in the unfolding are related by a translation, AND
- 2 there exists a path that “stays within” the unfolding, not hitting the vertex.

We will illustrate this using pictures in the next slide.

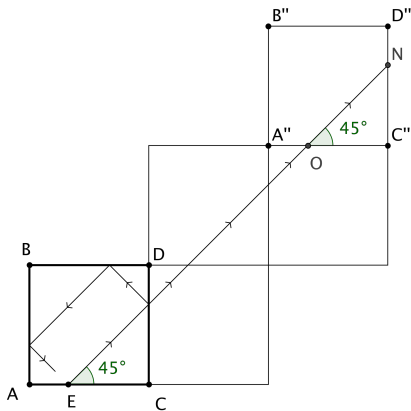
Let's look at unfolding of this periodic path:



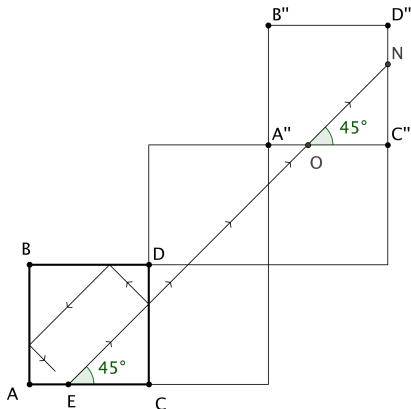
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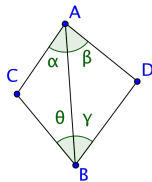


Since the line EN forms the same angle with the line AC and $A''C''$, so AC is parallel to $A''C''$, hence the last polygon is a translation of the first polygon. And the line EN lies “within” the unfolding.



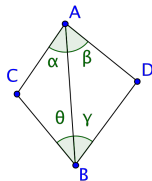
Space of Quadrilaterals

Q , the space of all quadrilaterals (up to similarity/rescaling) is characterized by 4 parameters.

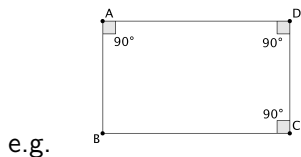


Space of Quadrilaterals

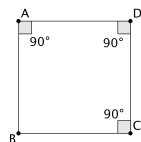
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3 parameters are not enough to characterize quadrilaterals.



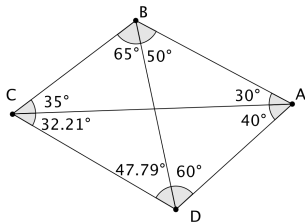
is not similar to



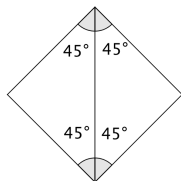
So, space of all quadrilaterals Q can be considered as a “subset” of \mathbb{R}^4 .
(although some elements might be represented by more than one elements in \mathbb{R}^4)

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e.g. $(65^\circ, 50^\circ, 60^\circ, 47.79^\circ)$ and $(30^\circ, 40^\circ, 32.21^\circ, 35^\circ)$ represent the same quadrilateral below.

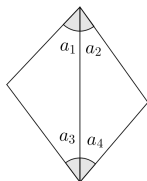


Near Square



The square is characterized by the coordinate $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$.

Near Square



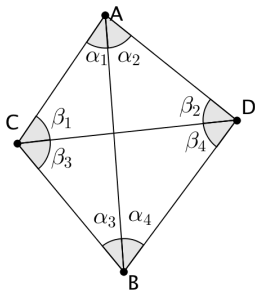
The square is characterized by the coordinate $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$.

Definition

A quadrilateral is ε -near square iff along one of the diagonal, $|a_i - \frac{\pi}{4}| < \varepsilon$ for $i = 1, \dots, 4$

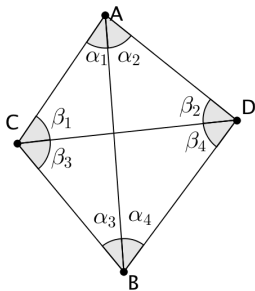
Lemma

In the diagram below, if $\varepsilon < \frac{\pi}{12}$, and $|\alpha_i - \frac{\pi}{4}| < \varepsilon$ for $i = 1, \dots, 4$, then $|\beta_j - \frac{\pi}{4}| < 3\varepsilon$ for $j = 1, \dots, 4$



Lemma

In the diagram below, if $\varepsilon < \frac{\pi}{12}$, and $|\alpha_i - \frac{\pi}{4}| < \varepsilon$ for $i = 1, \dots, 4$, then $|\beta_j - \frac{\pi}{4}| < 3\varepsilon$ for $j = 1, \dots, 4$



Consequence: Our definition of near-square is not arbitrary/overly affected by the choice of diagonal.

As long as one set of angles (e.g. α_i) stays near $\frac{\pi}{4}$, the other set of angles (e.g. β_j) will not stray too far from $\frac{\pi}{4}$.

1st Main Result

Theorem

If q is a quadrilateral that is $\frac{\pi}{107}$ -near square, then it has a periodic billiard path.

1st Main Result

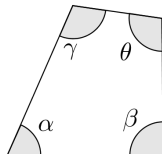
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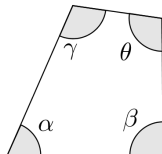
For every rectangle r , there exists an $\varepsilon_r > 0$, such that every quadrilateral that is ε_r near the rectangle r has a periodic billiard path.

Quick recap on how the proof works:
For every quadrilateral, if it is not a
rectangle, then choose any angle
 $\alpha < \frac{\pi}{2}$.



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Choose the smaller of the two adjacent angle, call it β .

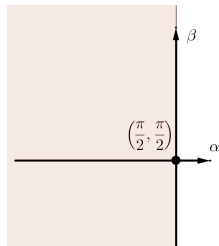
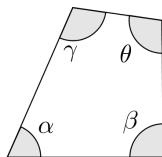


α - β plane

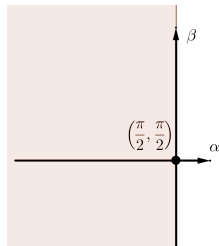
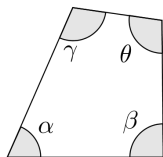
Quick recap on how the proof works:
For every quadrilateral, if it is not a rectangle, then choose any angle $\alpha < \frac{\pi}{2}$.

Choose the smaller of the two adjacent angle, call it β .

Every quadrilateral is represented by some point on the α - β left half plane and origin. ($\alpha < \frac{\pi}{2}$ or $\alpha = \beta = \frac{\pi}{2}$)

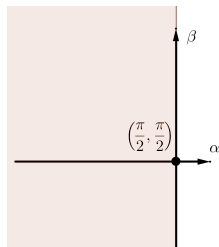
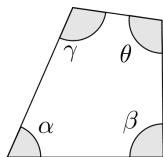


Remark: Being close to $(\alpha, \beta) = (\frac{\pi}{2}, \frac{\pi}{2})$ does not give you near square-ness. (e.g. the point $(\frac{\pi}{2}, \frac{\pi}{2})$ represents rectangles.)

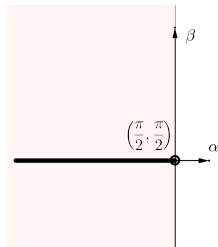


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So, we want to prove: Every point on this α - β plane, provided the quadrilateral they represent are $\frac{\pi}{107}$ near square, have periodic billiard path.

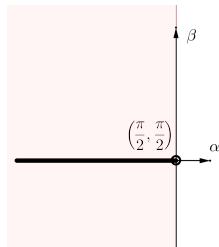


The idea: to chop up the α - β plane into different cases. For each case (assuming the near-squareness), try to find a periodic path that satisfy each case.



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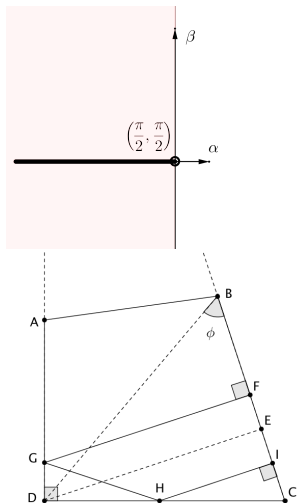
For example, in the bolded line in diagram to the right, it represents a right angle adjacent to an acute angle.



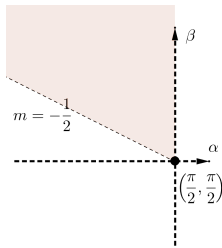
The idea: to chop up the α - β plane into different cases. For each case (assuming the near-squareness), try to find a periodic path that satisfy each case.

For example, in the bolded line in diagram to the right, it represents a right angle adjacent to an acute angle.

We can relate this case to billiard path in right triangle, as shown in the picture below.

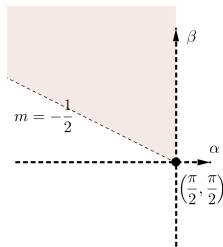


Another example: in the shaded region in diagram to the right, is represented by $\alpha + 2\beta > \frac{3}{2}\pi$.

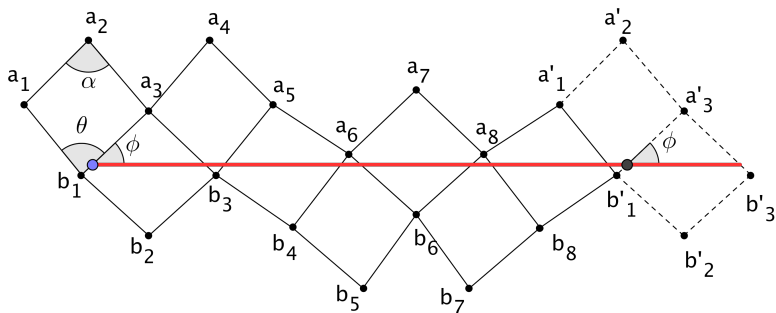


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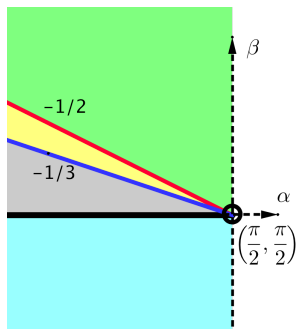
Which is to say: the angle opposite α is acute.



We found the following billiard path that satisfy the all quadrilateral in the shaded region, provided the quadrilateral is $\frac{\pi}{30}$ near square. We call it the "A" orbit.

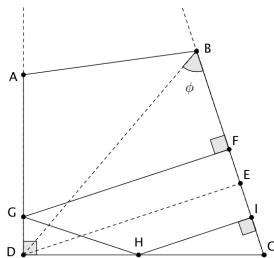


The eight regions that we considered are illustrated in the right: (4 regions, 3 lines, and origin).



Why so near square?

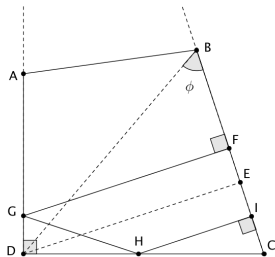
e.g. For the right angle-acute angle case, we just need the quadrilateral to be $\frac{\pi}{12}$ near square.



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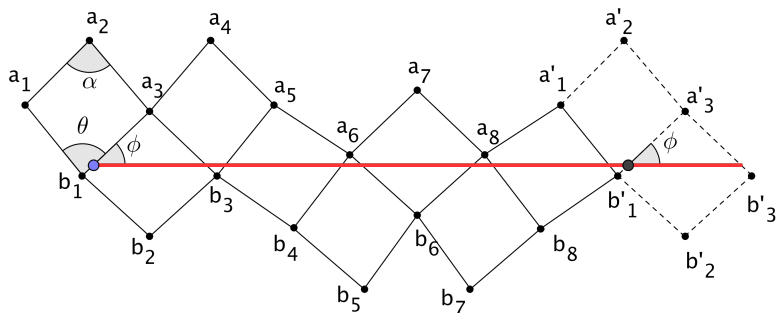
e.g. For the right angle-acute angle case, we just need the quadrilateral to be $\frac{\pi}{12}$ near square.

In particular, we need the angle ϕ to not exceed $\frac{\pi}{2}$



Why so near square?

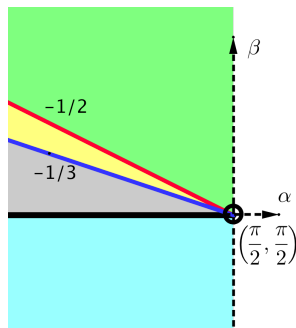
After some calculation, we can show that as long as the quadrilateral is $\frac{\pi}{30}$ near square, then we can “fit” the billiard path into the unfolding.



Why so near square?

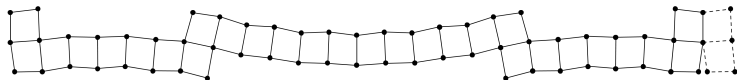
How near square do we need for each region (for a billiard path to exist)?
Below is a list of the “maximal tolerance”.

- Green: $\frac{\pi}{30}$
- Red: $> \frac{\pi}{107}$
- Yellow: $\frac{\pi}{107}$
- Blue: $> \frac{\pi}{56}$
- Gray: $\frac{\pi}{56}$
- Black: $\frac{\pi}{12}$
- Cyan: $\frac{\pi}{12}$
- Origin: $> \frac{\pi}{12}$



Why so near square?

The following is the unfolding of the periodic path we used to cover the yellow region.



Why so near square?

The following is the unfolding of the periodic path we used to cover the yellow region.

It is hard to “fit” a billiard path within the unfolding. Slight perturbation of the quadrilateral could potentially change the unfolding drastically, and hence unable to “fit” a periodic path within the unfolding.

