Bugs

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Escher's problem

- n ants on a Möbius band
- Ant 1 chases ant 2, ant 2 chases bug 3, ant 4 chases
- What happens?

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Lucas's problem

- Three bugs on the corners of an equilateral triangle and each one chases the next one at unit speed.
- What happens (Lucas, 1877)?

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Solution

Regular Hexagon

Regular Decagon

The general problem in Euclidean space

- Nonsymmetric configurations of n bugs in $\mathbb{R}^m$?
- Bugs sweep out

$$\{b_i : \mathbb{R}^+ \to \mathbb{R}^m\}_{i \in \mathbb{Z}/n}.$$

- Rule of motion

$$\dot{b}_i = \frac{b_{i+1} - b_i}{\|b_{i+1} - b_i\|}.$$

- When b_i catches b_{i+1}, they stay together
- Ends when all collide

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200 beetles

Typical behavior

- Beetles start out in a random configuration.
- They form a nice knot shape
- Knot shape undoes itself into a circular loop
- Circular loop contracts to a point

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100 beetles

1000 beetles

Finite time until collision

- Proposition: Beetles all collide in finite time.
- $L_i(t) \equiv d(b_i(t), b_{i+1}(t))$.
- $$\frac{d}{dt} L_i(t) = -1 + \cos(\theta_i),$$
  θ_i is $i-$th exterior angle of the piecewise geodesic path connecting the bugs
- Borsuk (1947):
  $$\sum_{i=1}^{n} |\theta_i| > 2\pi,$$
  so exists j for which $|\theta_j| \geq \frac{2\pi}{n}$.
- $$\frac{d}{dt}\left(\sum_{i=1}^{n} L_i(t)\right) = \sum_{i=1}^{n} \cos(\theta_i) - 1 \leq \cos(\theta_j) - 1 \leq \cos\left(\frac{2\pi}{n}\right) - 1,$$
  so process terminates in time less than or equal to $\frac{\sum_{i=1}^{n} L_i(0)}{1 - \cos(2\pi/n-1)}$.

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Asymptotic circularity and stable configurations

- Question: When does it happen that shape of bug loop is asymptotically circular as we approach collapse time?
- Partial answers from (Richardson 2001).
- The only invariant configurations are regular $n-$gons.
- The only locally attracting configurations are regular $n-$ gons for $n \geq 7$. (For $n < 7$, tend to get collapse to a line.)
- Question: What is basin of attraction of stable configurations?

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Bugs on manifolds

- *Manifold*: Space that looks locally like Euclidean space
- *Riemannian manifold*: Locally Euclidean space with extra structure for measuring lengths, angles, and volumes.
- Examples: Sphere, torus

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Geodesics

- Straight line segments, paths of least resistance, paths followed by physical particles.
- On manifolds embedded in $\mathbb{R}^n$, paths of zero tangential acceleration.
- Example: On a sphere, great circle arcs.
- *Closed* geodesics are smooth geodesic loops, periodic paths followed by physical particles.

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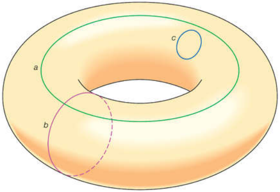
Bugs on compact manifolds

- Place consecutive bugs close enough together so that there's a unique geodesic connecting each one to the next
- Give each velocity equal to the unit tangent to the geodesic.
- Goal: understand what happens to the piecewise geodesic closed loop β_t connecting consecutive beetles $t \to \infty$
- Problem is interesting on manifold because not all loops are contractible
- Conjecture: Bug loops which do not contract to a point converge to closed geodesics.



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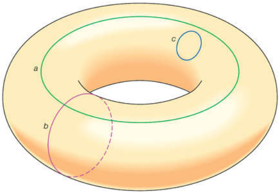

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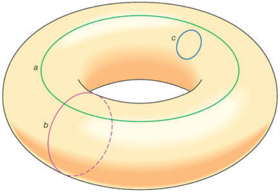

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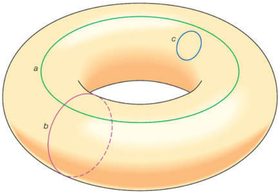

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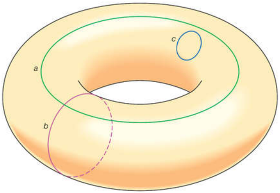

Subsequential convergence

- Proposition: Exists (t_j) going to ∞, and geodesic α so that $\beta_{t_j} \to \alpha$.
- $$L_i(t) \equiv d(b_i(t), b_{i+1}(t)),$$
  $$L(t) = \sum_i L_i(t).$$
- $$\frac{d}{dt} L(t) = \sum_i \left(-1 + \cos(\theta_i)\right),$$
  θ_i is $i-$th exterior angle of the piecewise geodesic path connecting the bugs
- Since $L_i(t)$ decreasing, bounded from below, exists subsequence $(t_j)_{j=1}^{\infty}$, $t_j \to \infty$ s.t. $\frac{d}{dt} L(t_j) \to 0$.
- Now, $\theta_i(t_j) \to 0$ for all i.
- Passing to subsequence, assume $b_i(t_j)$ converges, say to a_i. Let α be p.g. loop connecting a_i.
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Convergence in a special case

If there is a unique closed geodesic α that has a subsequence (β_{t_j}) converging to, then β_t converges to α.

- Assume for the sake of contradiction that β_t doesn't converge to α.
- Then there's an ε and a sequence $(t_j) \to \infty$ s.t. $d(\beta_{t_j}, \alpha) > \varepsilon$ for all t_j.
- Pass to subsequence so that β_{t_j} converges to a geodesic α'.
- $d(\alpha', \alpha) > \varepsilon$, so $\alpha \neq \alpha'$. Contradiction.

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Convergence for nonpositively curved manifolds

- Def: *Tubular ε-nhd* $N_\varepsilon(\alpha)$ of geodesic α is $\{p \in M | \inf_s d(p, \alpha) < \varepsilon\}$.
- Let α be a geodesic to which subsequence β_{t_j} converges.
- Fact: For ε small enough, $N_\varepsilon(\alpha)$ is *geodesically convex*: if $p, q \in N_\varepsilon(\alpha)$, so are shortest geodesics connecting p, q.
- Take any $\varepsilon > 0$ small enough that $N_\varepsilon(\alpha)$ is convex. Find T so that $\beta_T \subset N_\varepsilon(\alpha)$. By convexity, $\beta_t \subset N_\varepsilon(\alpha)$ for all $t > T$.

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On a torus

Convergence for unique local minimizers

- Closed geodesic α is a *unique local minimizer* of length if all closed loops in an nhd of α are longer.
- Proposition: If α is unique local minimizer and subsequence $\beta_{t_j} \to \alpha$, then $\beta_t \to \alpha$.
- Suffices to show that α is unique geodesic with subsequence of bug loops converging to it.
- Assume fsoc α' also has subsequence of bug loops converging to it. Since, $L(\beta_t)$ decreasing, $L(\alpha) = L(\alpha') \equiv l$.
- Since $L(\beta_t)$ decreasing, and there is a subsequence converging to each of α, α', there is for each $\varepsilon > 0$ a path of loops of length between l and $l + \varepsilon$ connecting α to α'. Call path of loops $b_\varepsilon : [0, 1] \to \text{Loops}(M)$.

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- Since $L(\beta_t)$ decreasing, and there is a subsequence converging to each of α, α', there is for each $\varepsilon > 0$ a path of loops of length between l and $l + \varepsilon$ connecting α to α'. Call path of loops $b_\varepsilon : [0, 1] \to \text{Loops}(M)$.

Convergence for unique local minimizers

- Choose δ so that $d(\gamma, \alpha) \leq \delta$ and $L(\gamma) = L(\alpha)$ implies γ is a reparameterization of α.
- For each ε, pick a time $s_\varepsilon \in [0, 1]$ so that $d(\beta_\varepsilon(s_\varepsilon), \alpha) = \delta$.
- By compactness, there is a sequence (ε_j), $j \to 0$, so that $\beta_{\varepsilon_j}(s_{\varepsilon_j})$ converges to some closed geodesic γ.
- By continuity of distance, $d(\gamma, \alpha) = \delta > 0$, and by sub-continuity of length $d(\gamma) < l$. Contradiction.

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On the projective plane

- Projective plane is upper half of sphere with opposite points on the equator glued together.

Connection to curve shortening flow?

- Beetle dynamics looks a lot like curve shortening flow
- Curve-shortening pushes each point on smooth curve in the direction of the curvature vector
- Curvature vector is direction in which curve turns
- Knot shape undoes itself into a circular loop
- Circular loop contracts to a point

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From Curtis McMullen's site

Something else to think about: Billiard bugs.