Bugs

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Escher’s problem

- *n* ants on a Möbius band
- Ant 1 chases ant 2, ant 2 chases bug 3, ant 4 chases
- What happens?
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Lucas’s problem

- Three bugs on the corners of an equilateral triangle and each one chases the next one at unit speed.
- What happens (Lucas, 1877)?
Lucas’s problem

- Three bugs on the corners of an equilateral triangle and each one chases the next one at unit speed.
- What happens (Lucas, 1877)?
Regular Hexagon
Regular Decagon
The general problem in Euclidean space

- Nonsymmetric configurations of \( n \) bugs in \( \mathbb{R}^m \)?
- Bugs sweep out:
  \[ \{ b_i : \mathbb{R}^+ \rightarrow \mathbb{R}^m \}_{i \in \mathbb{Z}/n}. \]
- Rule of motion:
  \[ \dot{b}_i = \frac{b_{i+1} - b_i}{\| b_{i+1} - b_i \|}. \]
- When \( b_i \) catches \( b_{i+1} \), they stay together.
- Ends when all collide.
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200 beetles
Typical behavior

- Beetles start out in a random configuration.
- They form a nice knot shape
- Knot shape undoes itself into a circular loop
- Circular loop contracts to a point
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100 beetles
1000 beetles
Finite time until collision

- Proposition: Beetles all collide in finite time.
- $L_i(t) \equiv d(b_i(t), b_{i+1}(t))$.

\[ \frac{d}{dt} L_i(t) = -1 + \cos(\theta_i), \]

$\theta_i$ is $i$–th exterior angle of the piecewise geodesic path connecting the bugs

- Borsuk (1947): \[ \sum_{i=1}^{n} |\theta_i| > 2\pi, \]

so exists $j$ for which $|\theta_j| \geq \frac{2\pi}{n}$.

\[ \frac{d}{dt} \left( \sum_{i=1}^{n} L_i(t) \right) = \sum_{i=1}^{n} \cos(\theta_i) - 1 \leq \cos(\theta_j) - 1 \leq \cos \left( \frac{2\pi}{n} \right) - 1, \]

so process terminates in time less than or equal to \[ \frac{\sum_{i=1}^{n} L_i(0)}{1 - \cos(2\pi/n - 1)}. \]
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Asymptotic circularity and stable configurations

- Question: When does it happen that shape of bug loop is asymptotically circular as we approach collapse time?
- Partial answers from (Richardson 2001).
  - The only invariant configurations are regular $n$–gons.
  - The only locally attracting configurations are regular $n$–gons for $n \geq 7$. (For $n < 7$, tend to get collapse to a line.)
- Question: What is basin of attraction of stable configurations?
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Question: What is basin of attraction of stable configurations?
Bugs on manifolds

- **Manifold**: Space that looks locally like Euclidean space
- **Riemannian manifold**: Locally Euclidean space with extra structure for measuring lengths, angles, and volumes.
- Examples: Sphere, torus
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Geodesics

- Straight line segments, paths of least resistance, paths followed by physical particles.
- On manifolds embedded in $\mathbb{R}^n$, paths of zero tangential acceleration.
- Example: On a sphere, great circle arcs.
- Closed geodesics are smooth geodesic loops, periodic paths followed by physical particles.
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- *Closed* geodesics are smooth geodesic loops, periodic paths followed by physical particles.
Bugs on compact manifolds

- Place consecutive bugs close enough together so that there’s a unique geodesic connecting each one to the next.
- Give each velocity equal to the unit tangent to the geodesic.
- Goal: understand what happens to the piecewise geodesic closed loop $\beta_t$ connecting consecutive beetles $t \to \infty$.
- Problem is interesting on manifold because not all loops are contractible.
- Conjecture: Bug loops which do not contract to a point converge to closed geodesics.
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Subsequential convergence

- Proposition: Exists \((t_j)\) going to \(\infty\), and geodesic \(\alpha\) so that \(\beta(t_j) \to \alpha\).

\[
L_i(t) \equiv d(b_i(t), b_{i+1}(t)),
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\[
L(t) = \sum_i L_i(t).
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\[
\frac{d}{dt} L(t) = \sum_i (-1 + \cos(\theta_i)),
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\(\theta_i\) is \(i\)-th exterior angle of the piecewise geodesic path connecting the bugs.

- Since \(L_i(t)\) decreasing, bounded from below, exists subsequence \((t_j)_{j=1}^\infty, t_j \to \infty\) s.t. \(\frac{d}{dt} L(t_j) \to 0\).

- Now, \(\theta_i(t_j) \to 0\) for all \(i\).

- Passing to subsequence, assume \(b_i(t_j)\) converges, say to \(a_i\). Let \(\alpha\) be a p.g. loop connecting \(a_i\).

- By continuity, \(\theta_i = 0\), so \(\alpha_i\) is a geodesic.
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Subsequent convergence

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- By continuity, \(\theta_i = 0\), so \(\alpha_i\) is a geodesic.
Convergence in a special case

If there is a unique closed geodesic \( \alpha \) that has a subsequence \((\beta_{t_j})\) converging to, then \( \beta_t \) converges to \( \alpha \).

- Assume for the sake of contradiction that \( \beta_t \) doesn’t converge to \( \alpha \).
- Then there’s an \( \varepsilon \) and a sequence \((t_j) \to \infty\) s.t. \( d(\beta_{t_j}, \alpha) > \varepsilon \) for all \( t_j \).
- Pass to subsequence so that \( \beta_{t_j} \) converges to a geodesic \( \alpha' \).
- \( d(\alpha', \alpha) > \varepsilon \), so \( \alpha \neq \alpha' \). Contradiction.
If there is a unique closed geodesic $\alpha$ that has a subsequence $(\beta_{t_j})$ converging to, then $\beta_t$ converges to $\alpha$.

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Def: Tubular $\varepsilon$-nhd $N_\varepsilon(\alpha)$ of geodesic $\alpha$ is $\{ p \in M | \inf_s d(p, \alpha) < \varepsilon \}$.

Let $\alpha$ be a geodesic to which subsequence $\beta_{t_j}$ converges.

Fact: For $\varepsilon$ small enough, $N_\varepsilon(\alpha)$ is geodesically convex: if $p, q \in N_\varepsilon(\alpha)$, so are shortest geodesics connecting $p, q$.

Take any $\varepsilon > 0$ small enough that $N_\varepsilon(\alpha)$ is convex. Find $T$ so that $\beta_T \subset N_\varepsilon(\alpha)$. By convexity, $\beta_t \subset N_\varepsilon(\alpha)$ for all $t > T$. 
Convergence for nonpositively curved manifolds

- **Def:** *Tubular* $\varepsilon$-*nhd* $N_\varepsilon(\alpha)$ of geodesic $\alpha$ is $\{p \in M | \inf_s d(p, \alpha) < \varepsilon\}$.

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On a torus
Convergence for unique local minimizers

- Closed geodesic $\alpha$ is a *unique local minimizer* of length if all closed loops in an nhd of $\alpha$ are longer.

- Proposition: If $\alpha$ is unique local minimizer and subsequence $\beta_{t_j} \to \alpha$, then $\beta_t \to \alpha$.

- Suffices to show that $\alpha$ is unique geodesic with subsequence of bug loops converging to it.

- Assume f.s.o.c $\alpha'$ also has subsequence of bug loops converging to it. Since, $L(\beta_t)$ decreasing, $L(\alpha) = L(\alpha') \equiv l$.

- Since $L(\beta_t)$ decreasing, and there is a subsequence converging to each of $\alpha, \alpha'$, there is for each $\varepsilon > 0$ a path of loops of length between $l$ and $l + \varepsilon$ connecting $\alpha$ to $\alpha'$. Call path of loops $b_\varepsilon : [0, 1] \to \text{Loops}(M)$.
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56
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Suffices to show that $\alpha$ is unique geodesic with subsequence of bug loops converging to it.

Assume fsoc $\alpha'$ also has subsequence of bug loops converging to it. Since, $L(\beta_t)$ decreasing, $L(\alpha) = L(\alpha') \equiv l$.

Since $L(\beta_t)$ decreasing, and there is a subsequence converging to each of $\alpha, \alpha'$, there is for each $\varepsilon > 0$ a path of loops of length between $l$ and $l + \varepsilon$ connecting $\alpha$ to $\alpha'$. Call path of loops $b_\varepsilon : [0, 1] \to \text{Loops}(M)$. 
Convergence for unique local minimizers

Choose $\delta$ so that $d(\gamma, \alpha) \leq \delta$ and $L(\gamma) = L(\alpha)$ implies $\gamma$ is a reparameterization of $\alpha$.

For each $\varepsilon$, pick a time $s_\varepsilon \in [0, 1]$ so that $d(\beta_\varepsilon(s_\varepsilon), \alpha) = \delta$.

By compactness, there is a sequence $(\varepsilon_j), j \to 0$, so that $\beta_{\varepsilon_j}(s_{\varepsilon_j})$ converges to some closed geodesic $\gamma$.

By continuity of distance, $d(\gamma, \alpha) = \delta > 0$, and by sub-continuity of length $d(\gamma) < l$. Contradiction.
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On the projective plane

- Projective plane is upper half of sphere with opposite points on the equator glued together.
Beetle dynamics looks a lot like curve shortening flow

- Curve-shortening pushes each point on smooth curve in the direction of the curvature vector
- Curvature vector is direction in which curve turns
- Knot shape undoes itself into a circular loop
- Circular loop contracts to a point
Connection to curve shortening flow?

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From Curtis McMullen’s site
Something else to think about: Billiard bugs.