Outer Billiards with Contraction

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Summary

• In section I, we discuss stability issues, shedding light on a striking difference between the regular pentagon and the regular septagon.
Summary

• In section II, we discuss trapezoidal outer billiards with contraction. We can prove that for any trapezoid there are infinitely many stable degenerate periodic orbits (SDPOs).
Summary

• In section III, we outline a proof that for certain choice of the polygon and the contraction, attracting Cantor sets exist.
# A Comparison

<table>
<thead>
<tr>
<th></th>
<th>Square / Triangle</th>
<th>Trapezoid</th>
<th>Reg. Pentagon</th>
<th>Reg. Septagon</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Finiteness</strong> in bounded ball</td>
<td>Yes</td>
<td>Yes (?)</td>
<td>No</td>
<td>No (?)</td>
</tr>
<tr>
<td><strong>Exotic Periodic Orbits</strong></td>
<td>None</td>
<td>Infinitely many</td>
<td>None (?)</td>
<td>???</td>
</tr>
<tr>
<td><strong>Stability</strong> of Periodic Domains</td>
<td>Yes</td>
<td>Yes (?)</td>
<td>Yes</td>
<td>No !</td>
</tr>
<tr>
<td>Possible Approach</td>
<td>Hierarchy of the tiling</td>
<td>Horizontal Slicing / quasihierarchy</td>
<td>Renormalization / quasihierarchy</td>
<td>???????</td>
</tr>
</tbody>
</table>
I. Stability
• Theorem. (Stability Criterion)
  • Given a periodic domain $Q$ for $P$, look at reflected images of $P$ determined by the combinatorics of $Q$. Then $Q$ is $\lambda$-stable if and only if the barycenter of all the images of $P$ lies in the interior of $Q$. 
Illustration of the Stability
Period 10 orbit
Period 30 orbit
• **Corollary.** *Symmetric* periodic domains are stable

• **Corollary.** *Odd* periodic domains are stable
• **Corollary.** When $n = 3, 4, 5, 6, 8$, all periodic domains for the regular $n$-gon are stable

• **Corollary.** If $P$ is centrally symmetric, all Culter periodic domains are stable
Stability of the Regular Pentagonal Periodic Domains
Let \((x, y, i)\) be the coordinates of the vertex \(\gamma\) as \(\gamma\) first enters an angle, and let \((x_1, y_1, i_1)\) be its coordinates when \(\gamma\) first enters the next angle. Define the map

\[
\phi: (x, y, i) \mapsto (x_1, y_1, i_1).
\]

Let the side of \(\gamma\) be equal to 1; define the shift \(S\) (in each angle) by

\[
S: (x, y, i) \mapsto (x + 3 + \sqrt{5}, y, i).
\]

**Lemma.** \(S \circ \phi = \phi \circ S\).

The proof is straightforward.
Figure 17

Regular Septagon: Unstable Pentagon

- Period 57848, Radius around 0.0001
Schwartz’s Zoo of Exotic Domains

- Diameter: 0.002, Period: up to 500000
II. Trapzoids
3-5-7 Conjecture.

- All the exotic periodic orbits have periods ending with either 3, 5, or 7.

- Finiteness follows.
3-5-7 Weaker Conjecture.

- For any positive integer congruent to either 3, 5, or 7, there exists a SDPO with that integer as its period.

- Infinitely many exotic periodic orbits!
3-5-7 Weaker Conjecture.

• Weaker conjecture is proved.

• Boils down to studying fixed points of orientation-reversing interval exchange transforms
The “Pinwheel” Dynamics
Interval Exchange Transform
III. Cantor Set
Affine Contractions

- Coloring Scheme by P. Hooper
Extending the Rotation Theory

• F. Rhodes, C. Thompson
Rotation number for monotone functions on the circle (1985)
Topologies and rotation numbers for families of monotone functions on the circle (1989)

• Summary
  – Well-defined and nice with respect to strictly increasing monotone maps of the circle
  – Continuous with respect to the Hausdorff metric on graphs
Extending the Rotation Theory (2)

• R. Brette
Rotation numbers of discontinuous orientation-preserving circle maps (2003)

• Summary
  – **If rational**, all orbits are asymptotically periodic.
  – **If irrational**, the limit set is either the whole circle or a unique Cantor set
Triangular Transition on Quadrilaterals
An Invariant Region

$\mathcal{L}(\epsilon)$
Dynamics of the Return Map
• Semiconjugacy
Proving existence of attracting Cantor set: **Ingredients**

- Rotation theory
- Bijection between periodic orbits
- Continuity of the rotation number
Attracting Cantor set Exists!
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• We also thank all ICERM staff.