Tripod Configurations

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Summer@ICERM 2013

August 8, 2013

Overview

Tripod Configurations in the Plane

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Definition 1

A tripod configuration of a closed plane curve γ consists of three normal lines dropped from γ meeting at a single point (the tripod center) and making angles of $\frac{2\pi}{3}$.

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Definition 2

A locally convex curve is a curve with nowhere vanishing curvature.

Existence of Tripod Configurations (Regular Polygons)

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Definition 3

A tripod configuration of a closed polygon P consists of three lines dropped from vertices of P meeting at a single point and making angles of $\frac{2\pi}{3}$ such that each line is normal to a support line of P at the vertex through which it passes.

Existence of Tripod Configurations (Regular Polygons)

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A tripod configuration of a closed polygon P consists of three lines dropped from vertices of P meeting at a single point and making angles of $\frac{2\pi}{3}$ such that each line is normal to a support line of P at the vertex through which it passes.

Theorem 4 (Summer@ICERM 2013)

A regular polygon with n vertices has n tripod configurations if $3 \nmid n$ and $\frac{n}{3}$ tripod configurations if $3 \mid n$.

Existence of Tripod Configurations (Plane Curves)

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Theorem 7 (Summer@ICERM 2013)

1. *If γ is a closed locally convex curve with winding number n, then γ has at least $2\lfloor\frac{n^2+2}{3}\rfloor$ tripod configurations.*
2. *Every plane curve has a tripod configuration.*

Proof of Theorem 7

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Given a triangle ABC, the largest equilateral triangle circumscribing it is its antipedal triangle with respect to its first isogonic center.

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Lemma 9

Let p, q, r be noncollinear points on a closed plane curve γ and let T be an equilateral triangle with each side passing through one of the three points. Then the equilateral triangle circumscribing γ with sides parallel to T is at least as large as T.

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Lemma 8

Given a triangle ABC, the largest equilateral triangle circumscribing it is its antipedal triangle with respect to its first isogonic center.

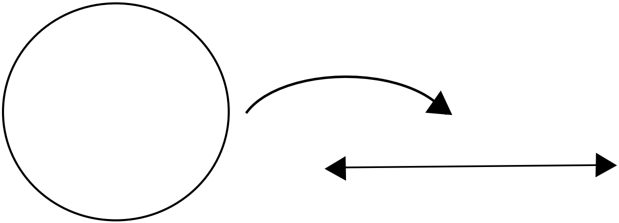
Lemma 9

Let p, q, r be noncollinear points on a closed plane curve γ and let T be an equilateral triangle with each side passing through one of the three points. Then the equilateral triangle circumscribing γ with sides parallel to T is at least as large as T.

Lemma 10

The largest equilateral triangle circumscribing a closed plane curve meets the curve exactly once per side.

Some Observations



- At least how many critical points does a smooth function on a circle have?

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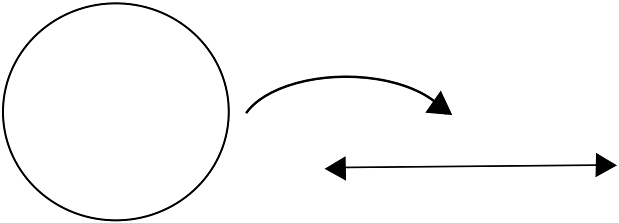
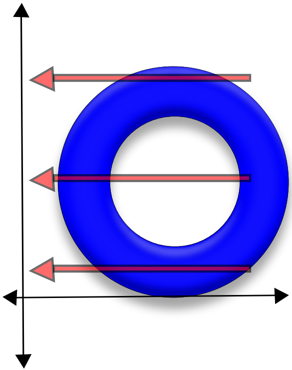
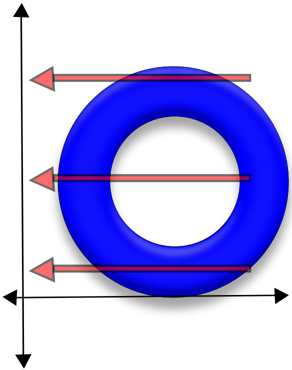


- At least how many critical points does a smooth function on a circle have?
- At least how many critical points does a smooth function on S^n have?



- Let $f : T^2 \to \mathbb{R}$ be the height function, or projection onto the z-axis.

The Classic Example



- Let $f : T^2 \to \mathbb{R}$ be the height function, or projection onto the z-axis.
- let $M^a = \{x \in T^2 \mid f(x) < a \in \mathbb{R}\}$.

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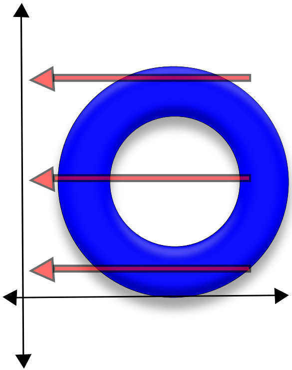
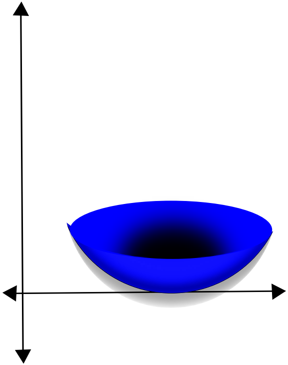


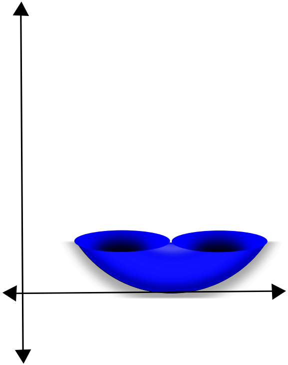
- Let $f : T^2 \to \mathbb{R}$ be the height function, or projection onto the z-axis.
- let $M^a = \{x \in T^2 \mid f(x) < a \in \mathbb{R}\}$.
- M^a is every point in T^2 below the height of a.

The Torus Continued...



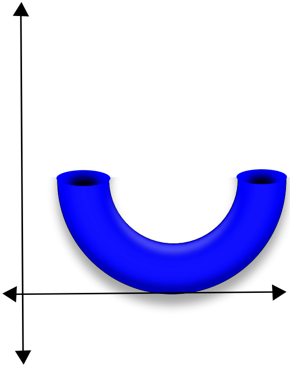
- The bottom is a critical point, a local minimum
- The index of the Hessian at the local minimum is zero
- M^{a_1} is homotopic to a 0-cell

The Torus Continued...



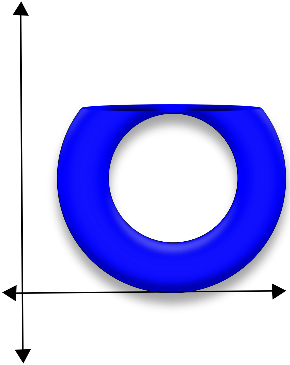
- The bottom point of the inner circle is a critical point, a saddle point
- The index of the Hessian at the point is one
- M^{a_1} is homotopic to a disk with a 1-cell attached.

The Torus Continued...



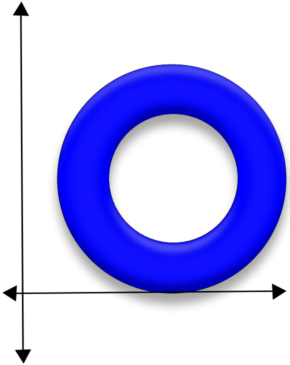
- We are not at a critical point of the function
- M^{a_3} is homotopic to M^{a_2}, there has been no change in the homotopy class.

The Torus Continued...



- The top point of the inner circle is a critical point, a saddle point
- The index of the Hessian at the point is one
- M^{a_4} is homotopic to a cylinder with a 1-cell attached

The Torus Continued...



- The top point of the torus is a critical point, a local maximum
- The index of the Hessian at the point is two
- M^{a_5} is homotopic to a punctured torus with a 2-cell attached

Basic Definitions

Definition 11 (non-degenerate)

A critical point of a function f is said to be *non-degenerate* if the Hessian of f at p is non-singular.

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Definition 12 (Morse index)

The *Morse index* of a critical point p of a function f is the index of the Hessian of f at p, i.e. the dimension of the largest subspace on which the Hessian is negative definite.

Important Results

Lemma 13 (The Morse Lemma)

If p is a non-degenerate critical point of f, then $\exists \phi$, a chart of M, such that $x_i(p) = 0 \forall i$ and $f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$ where k is the index of p.

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Theorem 14

Given $a < b$, if $f^{-1}[a, b]$ is compact and no critical values lie in the interval $[a, b]$ then M^a is a deformation retract of M^b.

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Theorem 14

Given $a < b$, if $f^{-1}[a, b]$ is compact and no critical values lie in the interval $[a, b]$ then M^a is a deformation retract of M^b.

Most importantly, this implies that M^a has the same homotopy class as M^b.

Important Results

The preceding lemma and theorem allow us to prove the following:

Theorem 15

If p is a non-degenerate critical point with Morse index k, $f(p) = a$ and then if we choose ϵ small enough so that $f^{-1}[a - \epsilon, a + \epsilon]$ is compact and contains no critical values other than p, then $M^{a+\epsilon}$ is of the homotopy class of $M^{a-\epsilon}$ with a k-cell attached.

The Morse Inequalities

- The preceding theorems show that the number of critical points is equal to the number of cells in the cell structure of the manifold defined by the function f.

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- This equality of the cell structure and the critical points can be used to prove certain inequalities about critical points from the homotopy class of a manifold.

The Morse Inequalities

Corollary 16

Let C^λ denote the number of critical points of f with Morse index λ.

$$\sum(-1)^\lambda C^\lambda = \chi(M)$$

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Corollary 17

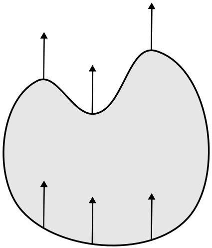
$$C^\lambda \geq b_\lambda(M)$$

Where $b_\lambda(M)$ is the λ'th Betti number of M.

Recent Results

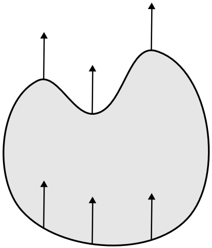
- Our work mainly uses results about Morse Theory on manifolds with boundary
- Francois Laudenbach published a paper on Morse Theory on manifolds with boundary in 2010
- He derives Morse inequalities using a chain complex defined on manifolds with boundary

Morse Theory on Manifolds with Boundary



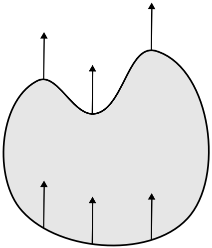
- Two new types of critical points appear on the boundary, when the gradient of f is normal to it.

Morse Theory on Manifolds with Boundary



- Two new types of critical points appear on the boundary, when the gradient of f is normal to it.
- Type N (Neumann)critical points occur when the gradient of f points inward along the boundary.

Morse Theory on Manifolds with Boundary



- Two new types of critical points appear on the boundary, when the gradient of f is normal to it.
- Type N (Neumann)critical points occur when the gradient of f points inward along the boundary.
- Type D (Dirichlet) critical points occur when the gradient of f points outward along the boundary.

Morse Theory on Manifolds with Boundary

We obtain new Morse inequalities from the following fact:

Theorem 18

Let C^λ and B^λ be the number of critical points of index λ in the interior and on the boundary (of D type) respectively. If $P(t)$ is the Poincare polynomial of M, $C(t) = \sum C^\lambda t^\lambda$, and $B(t) = \sum B^\lambda t^{\lambda+1}$, then:

$$B(t) + C(t) - P(t) = (1 + t)Q(t)$$

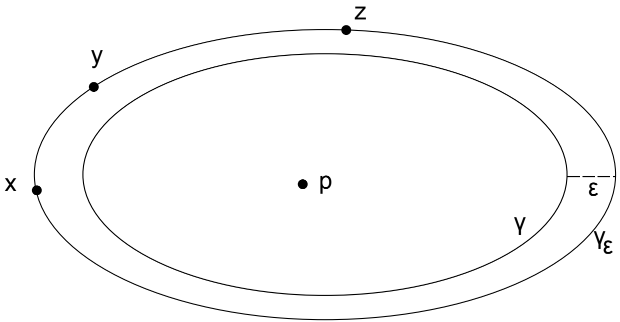
Where $Q(t)$ is a polynomial with non-negative integer coefficients.

Morse Theory and Tripod Configurations

We show that any convex curve in the plane and any convex curves sufficiently close to a circle in spherical and hyperbolic geometry have at least two tripod configurations.

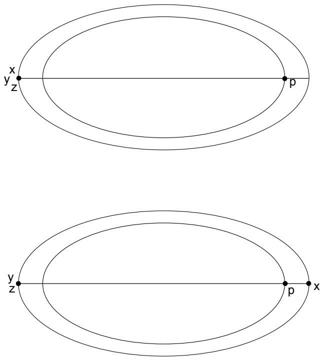
$$f : S^1 \times S^1 \times S^1 \times \overline{\mathbb{D}} \to \mathbb{R}, (x, y, z, p) \mapsto d(x, p) + d(y, p) + d(z, p)$$

Critical points of f where $p \in \mathbb{D}$ are tripod points.



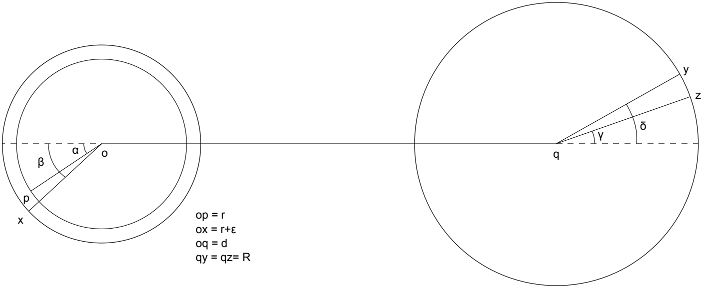
Morse Theory and Tripods - Planar Case

Parallel curves have the same tripod configurations. "Boundary" critical points of f when $p \in \partial\overline{\mathbb{D}}$ and the evolute is small:

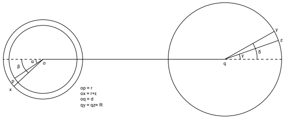


Morse Theory and Tripods - Planar Case

Use osculating circles to approximate the curve near "boundary" critical points of f to compute the Hessian up to second order approximation:



Morse Theory and Tripods - Planar Case



$$p = (-r\cos\alpha, -rsin\alpha) \approx (-r(1-\frac{\alpha^2}{2}), -r\alpha)$$

$$x = (-(r+\epsilon)\cos\beta, -(r+\epsilon)\sin\beta) \approx (-(r+\epsilon)(1-\frac{\beta^2}{2}), -(r+\epsilon)\beta)$$

$$y = (d + R\cos\gamma, R\sin\gamma) \approx (d + R(1-\frac{\gamma^2}{2}), R\gamma)$$

$$z = (d + R\cos\delta, R\sin\delta) \approx (d + R(1-\frac{\delta^2}{2}), R\delta).$$

$$\begin{pmatrix} \frac{r(r+\epsilon)}{\epsilon} - \frac{2r(d+R)}{d+R+r} & -\frac{r(r+e)}{e} & \frac{Rr}{d+R+r} & \frac{Rr}{d+R+r} \\ -\frac{r(r+e)}{e} & \frac{r(r+e)}{e} & 0 & 0 \\ \frac{Rr}{d+R+r} & 0 & -\frac{R(d+r)}{d+R+r} & 0 \\ \frac{Rr}{d+R+r} & 0 & 0 & -\frac{R(d+r)}{d+R+r} \end{pmatrix}$$

Theorem 19

Given an $n \times n$ matrix A, call the determinant of the $i \times i$ upper-left corner the ith leading minor and denote it by d_i. Assume that A is symmetric and the d_i's are non-zero...Then $d_1, d_2/d_1, d_3/d_2, ..., d_n/d_{n-1}$ are diagonal entries in a diagonalization of A.

Morse Theory and Tripods - Planar Case

Morse indices

Case 1:

$$\begin{cases} 4, & d > 0 \\ 3, & d < 0 \end{cases}$$

Case 2:

$$\begin{cases} 3, & d > 0 \\ 2, & d < 0 \end{cases}$$

Morse Theory and Tripods - Planar Case

Theorem 18

Let C^λ and B^λ be the number of critical points of index λ in the interior and on the boundary (of D type) respectively. If $P(t)$ is the Poincare polynomial of M, $C(t) = \sum C^\lambda t^\lambda$, and $B(t) = \sum B^\lambda t^{\lambda+1}$, then:

$$B(t) + C(t) - P(t) = (1+t)Q(t)$$

Where $Q(t)$ is a polynomial with non-negative integer coefficients.

A convex curve has as many "short" diameters ($d < 0$) as "long" diameters ($d > 0$). Let m be the number of "short" ("long") diameters.

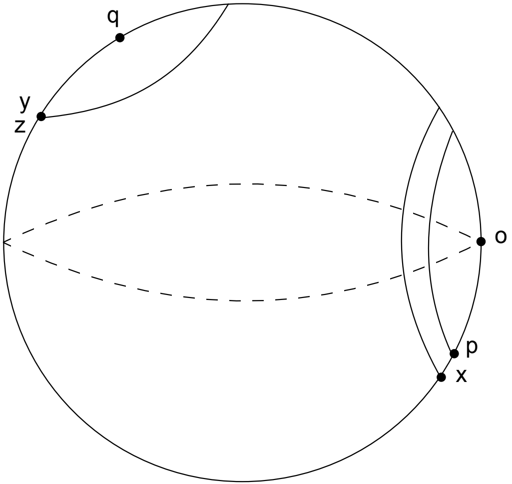
$$\begin{aligned} 2mt^5 + 6mt^4 + 2mt^4 + 6mt^3 + C(t) - (1+t)^3t^2 &= (1+t)Q(t) \\ (1+t)(2mt^4 + 6mt^3) + C(t) - (1+t)^3t^2 &= (1+t)Q(t) \\ 6(1+t) &| C(t) \end{aligned}$$

So the curve has at least 2 tripod configurations.

Morse Theory and Tripods - Spherical Case

The "small evolute" condition becomes a "close to a circle" condition on the sphere. Use a second order approximation with geodesic circles.

op = r
ox = r+ε
oq = d
qy = qz = R



Morse Theory and Tripods - Spherical Case

Morse indices (same as in planar case)
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$$\begin{cases} 4, & d > 0 \\ 3, & d < 0 \end{cases}$$

Case 2:

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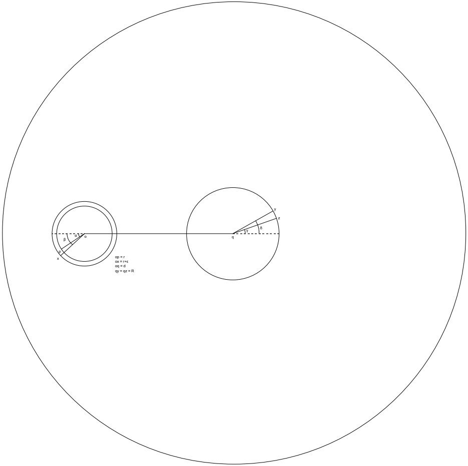
Morse Theory and Tripods - Hyperbolic Plane Case

Use a second order approximation with geodesic circles in the Poincaré disc model.



Morse Theory and Tripods - Hyperbolic Plane Case

Morse indices (same as in planar case)

Case 1:

$$\begin{cases} 4, & d > 0 \\ 3, & d < 0 \end{cases}$$

Case 2:

$$\begin{cases} 3, & d > 0 \\ 2, & d < 0 \end{cases}$$