Computational Dynamics and Topology

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Focus of the program: The main focus is the study of mathematical problems that originated from applications in physics and engineering. The mathematics involves applied topology and dynamical systems. Many of the sample problems will require numerical experiments which can lead to better formulations and may help finding the solutions.

1 Applied Topology

The goal of these projects is to develop intuition and understand the topology of non-traditional configuration spaces that arise in applications.

Locking configuration of hard balls (disks) in box (square) The archetypal dynamical system of statistical physics, the hard frictionless balls in a box (*Boltzmann gas*) has been studied for a century. Yet the basic question about the topology of these spaces – are these spaces connected? – is unanswered, in the physically interesting and important *gaseous* regime (when $nr \to \infty$, but $nr^2 \to 0$). In principle, disconnected configuration spaces of this kind exist (consider a circular bearing-like ring in a round container). Do such configurations exist in a polygonal box?

More general questions related to the topology of the space of non-overlapping disks configurations are abound, often leading to elementary (yet hard!) problems about existence of equilibrium configurations of touching disks in a box. Here equilibrium means reactive forces can be assigned to each contact between the touching disks (or disks and the boundary), so that the the total force acting on each disk vanishes. The so-called Böröczky's bridge might be a useful building block...

References

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[2] K. Böröczky. Über stabile Kreis- und Kugelsysteme. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 7 (1964), 79–82.

[3] R. Connelly. Rigidity of packings. European J. Combin. 29 (8) (2008), 1862–1871.

Tangle configuration spaces *Tangles*, a popular toy, consists of a chain of $\pi/2$ long arcs of unit circle connected so that the resulting loop is differentiable. A simple count suggests that the dimension of the configuration space of a chain of n arcs (mod Euclidean motions) is $n-5$. Surprisingly, perhaps, the configuration space of 6-chains is disconnected: one component indeed is a circle, the other is a point (these component correspond to Bricard and convex octahedra, correspondingly). What is the general situation? What are topologies for the general n, perhaps for generic arc lengths?

There exists a natural isometry between the space of tangles and certain moduli spaces of hinged bands, whose study was initiated by Streunu et al - but the theory of hinged structures is in its infancy as well, with most questions unresolved...

References

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Topology of spaces of coverings Consider a (compact) metric space X, and a number $r > 0$. We say that $\{x_1, \dots, x_N\}$ is an *r-covering* (or r-net) if the unions of r-balls around x_i covers X. The topology of the spaces of coverings is important in engineering applications, but its study has barely begun. Baryshnikov established that the space of coverings of the unit intervals by subintervals of lengths $2r$ has the homotopy type of k-skeleton of the the permutahedron P_n: here k is defined as the *excess*: the larges number k for which $(n-k)$ r-balls still cover X.

This implies that the space of coverings of an interval has the homotopy type of a wedge of spheres.

Next natural question to address deals with multiple coverings: what is the topology of the space of point configurations such that each point of X is covered by at least two r-balls?

2 Applied Computational Dynamics

Hamiltonian dynamics is a classical field that can be traced back to the discovery by Newton of the laws of motion of celestial bodies. The main goal is to understand the behavior of solutions both qualitatively and quantitatively. Besides physics related systems, Hamiltonian dynamics arises naturally in optimization problems such as the linear search problem and also in the so-called switching systems where energy is preserved. The theory of switching systems is now an active area of engineering research, see [1].

Search Problem Search theory appeared in WW2 when Bernard Osgood with his colleagues developed techniques to search U-boats. Later Bellman and Beck formulated a linear search problem.

Consider a searcher looking for an object that is hidden on the line with some known probability distribution. He starts at the origin and moves certain distance $x_1 > 0$ to the right, then turns around and moves to the left to $-x_2 < 0$, then turns right and moves to $x_3 > x_1$ and so on until the object is found. The natural question posed by Bellman and Beck is what is the optimal sequence of turning points, see *e.g.* [1].

It turns out that an optimal sequence is an orbit of two dimensional mappings. These maps appear to be integrable for a large class of probability distributions. The maps take the form

$$(x, y) \to (-y/f'(x), f(x)),$$

where $f(x)$ is a cumulative distribution function. Some of these maps exhibit the so-called integrable dynamics. The most simple example is the map with generating function given by

$$h(x_1, x_2) = x_2^2(1 - x_1)^2.$$

The map itself is given implicitly by

$$y_1 = -\frac{\partial h}{\partial x_1} \qquad y_2 = \frac{\partial h}{\partial x_2}$$

The goal of this project is to investigate integrability numerically and try to understand analytically when it happens.

References

[1] Y. Baryshnikov, V. Zharnitsky. Search on the brink of chaos, Nonlinearity 25 (2012), 3023–3047.

[2] A. Beck, On the linear search problem, Israel J. Math. 2 (1964), 221–228.

[3] R. Bellman, Research Problem No. 63-9, SIAM Review 5 (1963), 274.

Cyclic evasion systems The problem of cyclic pursuit or $n-$bug problem is a classical one, see, *e.g.* an article by Klamkin and Newman [2]. Because of applications in robotics and availability of more powerful computers, there has been recently some revival of interest in the n-bug problem and various generalizations.

The related problem of cyclic evasion, where each bug runs with the unit velocity directly away from one other bug, corresponds to reversing the time. This problem has also received some attention, especially in the computer science literature. It was observed numerically and verified with heuristic arguments that asymptotically, the n bug configuration converges either to a regular (convex or star) polygon or to a line configuration. In particular, a complete and short proof in this simplest case of 3 bugs can be found in [1].

This project will deal with the case $n = 4$. To fix the notation, let each bug be represented by $\mathbf{r_i} = (x_i, y_i) \in \mathbb{R}^2, i = A, B, C, D$. The bugs move according to

$$\dot{\mathbf{r_i}} = \frac{\mathbf{r_i} - \mathbf{r_{i+1}}}{|\mathbf{r_i} - \mathbf{r_{i+1}}|},$$

where $i + 1$ is understood as cyclic shift $A \to B \to C \to D \to A$.

Using elementary geometry, one obtains the following equations, see, *e.g.* [2] for the angles and side lengths:

$$\begin{aligned} \dot{l}_i &= 1 + \cos\alpha_i, \\ \dot{\alpha}_i &= \frac{\sin\alpha_{i-1}}{l_{i-1}} - \frac{\sin\alpha_i}{l_i}, \end{aligned} \tag{1}$$

The goal is to find what is the limiting shape of the quadrilateral: line configuration, a square, or a bow-tie.

References

[1] M. Arnold, V. Zharnitsky. Cyclic evasion in the three bug problem, to appear in American Math. Monthly, 2015.

[2] M. Klamkin, D. Newman. Cyclic pursuit or “the three bugs problem”, American Math. Monthly 78 (1971), 631–639.

Muti-agent control problem One of the most popular tasks in control systems is to arrange for a number of agents (or robots) to rendezvous at a particular location. In many applications the agents can only interact with each other. Given an arbitrary initial conditions, can they all arrive at a single location? In a more formal setting, in a certain scenario, the question is: can one stabilize a fixed point in the system of equations

$$\dot{x} = f(x),$$

i.e., make all solutions converge to a fixed point, say $x = 0$. Of course, in the absence of any restriction on f, one can take $f_i(x) = -x_i$.

But now, suppose some agents cannot communicate with each other. For example, agent 1 knows its own location and that of agent 2. Agent 2 knows location of agent 3 and agent 3 knows location of agent 1. Can they rendezvous under such restriction? Our system becomes

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_3) \\ \dot{x}_3 &= f_3(x_1). \end{aligned}$$

We hope to do numerical exploration to see if stabilization is possible. Note that if we consider only linear functions f_1, f_2, f_3 then a simple analysis of characteristic polynomial shows that stabilization is impossible, see [1] for a substantial treatment of stability of sparse linear systems. Therefore, one needs to search the answer among strongly nonlinear vector fields.

References

[1] M.-A. Bellabas, Sparse stable systems. Systems and Control Lett. 62 (2013), 981–987.

Pinball system: energy growth in switching Hamiltonian systems Consider a particle bouncing on the line between two stationary walls. One of the walls increases (decreases) velocity of the particle by 1 when the particle hits the wall at $t \in [2n, 2n+1] (t \in [2n+1, 2n+2])$. Interaction with the other wall does not change the particle's velocity. Such system is representative one in the switching systems but it is also related to classical stability theory in mechanics.

The basic question is to understand if the velocity (energy) can grow without bound in this system and at what rate. In a recent preprint [1], a solution with unbounded energy growth was constructed for a special case. The goal of this project will be to extend this construction to several other relevant problems.

References

[1] M. Arnold, V. Zharnitsky. Pinball dynamics: unlimited energy growth in switching Hamiltonian systems, preprint 2013.

3 PDE models

Modeling crowd dynamics The goal of this project is to develop a formulation that would model stampede phenomenon in the case of saturated crowd (individual pedestrians cannot change their position in the crowd). The statistical models have been considered earlier, see, *e.g.* [1]. The actual numbers can be found in [3]. In the case of a dense crowd, fluid dynamics models, such as those used for traffic modeling, have been employed, see [2].

Humans can produce forces, *e.g.*, by pushing with their hands. This can be modeled by considering system of particles with some interaction law, such as nonlinear elastic force, may be with delayed response.

One of the central goals in this field is to predict when dangerous pressures can form in the crowd by using mathematical models. Among recent works, see, *e.g.*, paper by Hughes [4]. This article develops a general model to study flow of pedestrians.

One of the specific goals in this project is to consider simpler crowd configurations such as one dimensional line or two-dimensional square. Use available model (or modify them) to explain how external (boundary) conditions with some initial conditions (too much density) can cause large (live dangerous) pressure inside the domain.

References

[1] D. Helbing, I. Farkas, T. Vicsek. Simulating dynamical features of escape panic. Nature, Vol 407, 2000.

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[3] Safer Crowds website, http://www.safercrowds.com/CrowdDisasters.html

[4] R. Hughes. A continuum theory for the flow of pedestrians. Transportation Research Part B 36 (2002), 507–535.

Non-smooth geodesics in sub-Riemannian geometry This project requires some knowledge of differential geometry. Consider sub-Riemannian structure in $\mathbb{R}^3$ with the metric

$$ds^2 = dx^2 + dy^2,$$

i.e., traveling in z-direction does not affect the distance. Assume there is a two-dimensional plane distribution given by the differential one-form

$$\omega = dz - B(x, y)dx.$$

We call piecewise smooth curves $\gamma(t) = (x(t), y(t), z(t))$ geodesics if they satisfy the differential relation $\dot{z} = B(x, y)\dot{x}$ and minimize length functional

$$\int_\gamma \sqrt{dx^2 + dy^2}.$$

The set where the form ω and $d\omega$ are linearly dependent ($\omega \wedge d\omega = 0$) is called singular, and outside of it, all geodesics are smooth. To try to construct an example of non-smooth geodesic, we let

$$B(x, y) = y^3/3 - \epsilon^2 x^2 y,$$

where ϵ is a small parameter. Note that

$$\omega \wedge d\omega = B_y dx \wedge dy \wedge dz,$$

and singular set is given by

$$B_y = y^2 - \epsilon^2 x^2 = (y - \epsilon x)(y + \epsilon x) = 0.$$

The wedge given by two pieces of the straight lines $y = \pm\epsilon x, y \geq 0$ is a candidate for non-smooth geodesic. It turns out, that it suffices to check that all smooth geodesics are longer than the hypothetical non-smooth one.

From variational calculus, it follows that smooth (normal) geodesics satisfy a system of ODEs

$$\begin{aligned} \ddot{x} &= - \lambda\, B_y(x, y)\, \dot{y} \\ \ddot{y} &= \lambda\, B_y(x, y)\, \dot{x}. \end{aligned}$$

The self-contained problem is to show that *any solution of the ODEs connecting two points on the wedge is longer that the corresponding part of the wedge*.

First step is to study the solutions of ODEs with $\epsilon = 0$. It is an integrable Hamiltonian system and it might even have explicit solution.

Next step is to consider perturbation of solutions for small ϵ using various asymptotics in ϵ and even computational techniques. Our hope is that for small ϵ one can estimate length of the solutions.

References

[1] D. Liberzon, Switching in systems and control. Springer 2003.

[2] R. Montgomery, A tour of Subriemannian Geometries, Their Geodesics and Applications. AMS 2002.