The Topology of Configuration Spaces of Coverings

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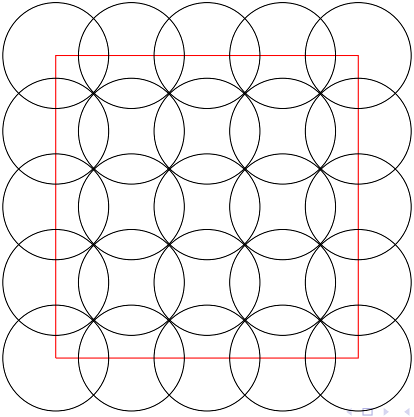
Overview

Introduction

General Question

Question

Given a metric space Y, a radius r and n closed balls of this radius, what is the topology of the configuration space of the balls (i.e. their centers) such that every point in Y is covered by (at least) one ball?



Configuration Spaces

Given n balls, label these balls $1, 2, \ldots, n$. Suppose $Y \subset \mathbb{R}^d$, and consider all vectors in $Y^n \subset \mathbb{R}^{dn}$ of the form $\vec{x} = (\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n})$, where ball i has center $\vec{x_i}$.

Definition (Configuration Space)

The *configuration space* of coverings of Y is all $\vec{x} \in Y^n$ such that Y is covered, i.e.

$$\mathrm{Cov}_n(r, Y) = \{\vec{x} \in Y^n \mid \forall y \in Y \; \exists \; 1 \le i \le n \text{ s.t. } d(y, x_i) \le r\}$$

Single coverings of the interval

Suppose we now consider coverings of the unit interval $I = [0, 1]$.

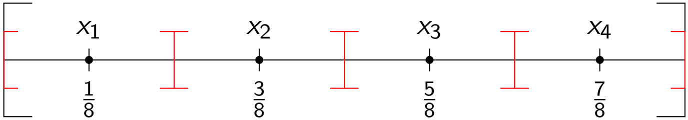


Figure: The configuration above corresponds to the point $(\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}) \in \mathbb{R}^4$

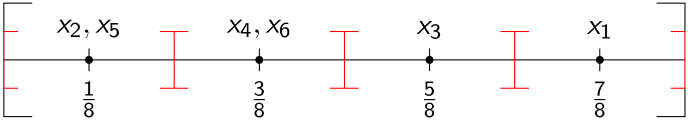


Figure: The configuration above corresponds to the point $(\frac{7}{8}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{1}{8}, \frac{3}{8}) \in \mathbb{R}^6$

Definition (Excess)

The excess given a radius r is defined as the largest number m for which it is still possible to cover the interval with $(n-m)$ r-balls.

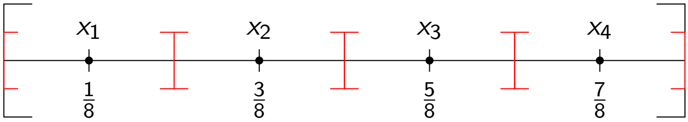


Figure: Excess 0, as with 4 balls of radius $\frac{1}{8}$, we can just cover I.

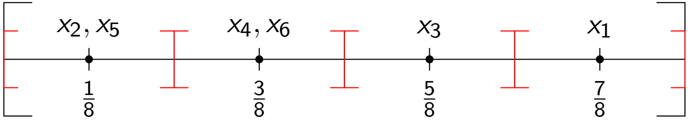


Figure: Excess 2, as we can cover the interval with 4 balls of radius $\frac{1}{8}$.

Background and Goal

Theorem (Baryshnikov)

$Cov_n(r, I) \sim Skel_m(P_n)$, where m is the excess, $Skel_m$ is an m-skeleton, and P_n is the permutahedron on n vertices.

Example ($n = 3$; the 3-permutahedron is a 2-dimensional hexagon)

- for $0 \leq 2r < \frac{1}{3}$, cannot cover, so $\mathrm{Cov}_3(r, I) \cong \emptyset$
- for $\frac{1}{3} \leq 2r < \frac{1}{2}$, $m = 0$, $\mathrm{Cov}_3(r, I) \sim$ vertices of hexagon (0-sk.)
- for $\frac{1}{2} \leq 2r < 1$, $m = 1$, $\mathrm{Cov}_3(r, I) \sim$ 1-sk. of hexagon $\sim S^1$
- for $1 \leq 2r$, $m = 2$, contractible

Our Goal

Find an analogue for the case of k-covering I, where k is arbitrary.

Indices

Definition

Define "indices" as points in the unit interval of the form $i_j = \frac{2j-1}{2n}$ for $1 \leq j \leq n$.

Suppose we have balls of radius $r = \frac{1}{2n}$. Then if we are k-covering I, kn balls will cover I, and then the excess $m =$(# of balls)$-kn$.

k-coverings of the unit interval

The Space of Double Coverings: Excess 1

Suppose now that we want to double-cover every point in I, so that

$$\forall\ y \in I,\ \exists\ 1 \leq j \neq k \leq 2n+1 \text{ s.t.} \max(d(y, x_j), d(y, x_k)) \leq r$$

Definition

Let $2\text{-Cov}_{2n+1}(r, I)$ be the configuration space of double coverings of the interval with $2n+1$ balls, with $\frac{1}{2n} \leq r < \frac{1}{2(n-1)}$, so the excess is 1.

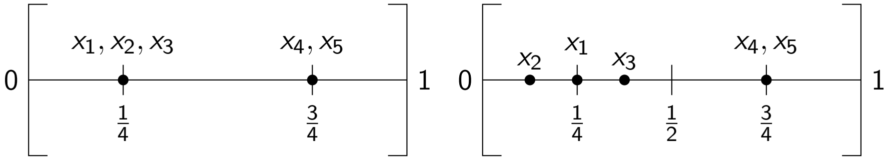
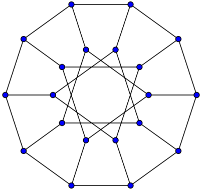


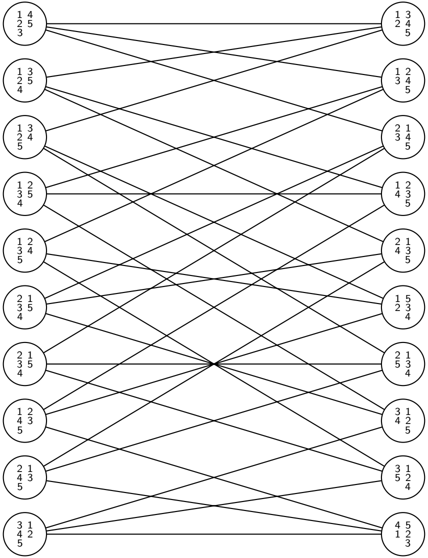
Figure: Two configurations with 5 balls of radius $\frac{1}{4}$ which double-cover I, corresponding to the the points $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})$ and $(\frac{1}{4}, \frac{1}{8}, \frac{3}{8}, \frac{3}{4}, \frac{3}{4})$, respectively, in $2\text{-Cov}_5(\frac{1}{4}, I)$.

The $n = 5$ case and the Desargues Graph

Theorem

For $\frac{1}{4} \leq r < \frac{1}{2}$ (excess 1), 2-$Cov_5(r, I) \cong_h G(10, 3)$, the "Desargues Graph"- the bipartite double cover of the Petersen Graph $G(5, 2)$.





Theorem 1: Our space is homotopic to a graph

Theorem

For $r = \frac{1}{2n}$ (excess 1), 2-$Cov_{2n+1}(r, I) \sim G$, for G a graph, i.e. a 1-dimensional simplicial complex.

Definition

Let $G_{2,n} \subset$ 2-$\text{Cov}_{2n+1}(r, I)$ be the following graph. For $\vec{x} \in$-$\text{Cov}_{2n+1}(r, I)$ to be in $G_{2,n}$ we first of all require that $\forall\ 1 \leq j \leq n, \exists\ 1 \leq p \neq q \leq 2n+1$ s.t. $i_j = x_p = x_q$. Thus, any point on this graph has at least 2 balls centered at each index i_j. A vertex of this graph also has one index with 3 balls centered at it. An edge of this graph has exactly 2 balls centered at each index, and one ball centered in an interval of the form $(i_j, i_{j+1}) = (\frac{2j-1}{2n}, \frac{2(j+1)-1}{2n})$ for $1 \leq j \leq n-1$.

Some Lemmas

Theorem

For any double-covering in 2-$Cov_{2n+1}(\frac{1}{2n}, I) \subset I^{2n+1}$*, every index must have at least* 1 *ball centered at it, that is:*

$$\forall\ 1 \leq j \leq n,\ \exists\ 1 \leq k \leq 2n+1 \text{ s.t. } i_j = x_k.$$

Some Lemmas

Theorem

For any double covering, at most 1 ball can be centered in any interval of the form (i_j, i_{j+1}) for $1 \leq j \leq n-1$.

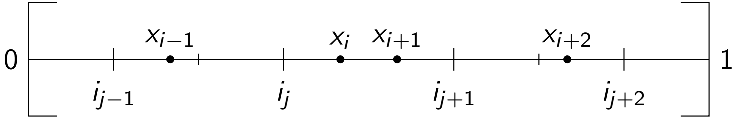


Figure: The above cannot happen.

Theorem

Suppose a double-covering has no balls centered in $(0, i_1)$ or $(i_n, 1)$. Suppose the balls with centers $x_{I_1}, x_{I_2}, \dots, x_{I_p}$ (re-labeled in ascending order) are not centered at indices, for $1 \leq I_j \leq 2n+1$ and $1 \leq p \leq n+1$. Then $x_{I_1} \in (i_j, i_{j+1}), \dots, x_{I_p} \in (i_{j+p-1}, i_{j+p})$ for $1 \leq j \leq n$.

Figure: The above cannot happen.

Flow for the Space of Double Coverings

$n = 3$, excess 1; 7 balls of radius $\frac{1}{6}$

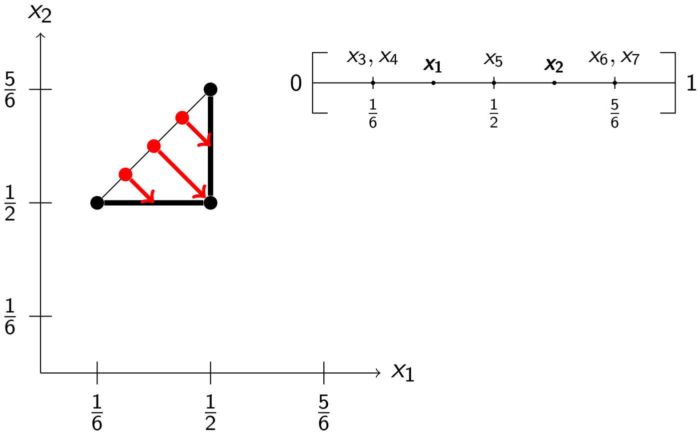


Figure: $\frac{1}{6} \leq x_1 \leq \frac{1}{2} \cap \frac{1}{2} \leq x_2 \leq \frac{5}{6} \cap x_2 - x_1 \leq \frac{1}{3}$

Flow for the Space of Double Coverings

$n = 4$, excess 1; 9 balls of radius $\frac{1}{8}$

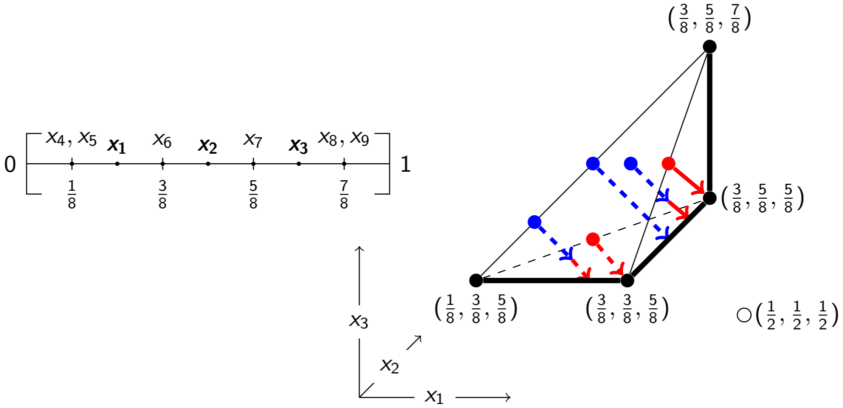


Figure: $\frac{1}{8} \leq x_1 \leq \frac{3}{8} \cap \frac{3}{8} \leq x_2 \leq \frac{5}{8} \cap \frac{5}{8} \leq x_3 \leq \frac{7}{8} \cap x_2 - x_1 \leq \frac{1}{4} \cap x_3 - x_2 \leq \frac{1}{4}$

Main Theorem

We calculated the vertex and edge counts of $G_{k,n}$:

$$V(G_{k,n}) = k\binom{kn+1}{k\ k\ k\ k\ \ldots\ k+1} = \frac{k(kn+1)!}{(k!)^{n-1}\cdot(k+1)}$$

$$E(G_{k,n}) = \frac{\frac{2}{n}\cdot V\cdot(k+1)+\frac{n-2}{n}\cdot V\cdot 2(k+1)}{2} = \frac{V(n-1)(k+1)}{n} = \frac{k(kn+1)!(n-1)}{n(k!)^{n-1}}$$

Theorem (See, for example: [Katok, 2006])

Suppose X and Y are 1-dimensional simplicial complexes, i.e. graphs. Then $X \sim Y \leftrightarrow \chi(X) = \chi(Y)$, where $\chi(X) = V(X) - E(X)$, where $\chi(X)$ is called the "Euler Characteristic" of X.

Theorem (Main Theorem)

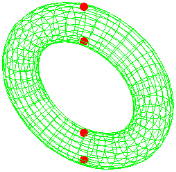
k-$Cov_{kn+1}(\frac{1}{2n}, I) \cong_h G_{k,n}$, with the above vertex and edge counts.

Future Work: Extending to higher excesses

Conjecture

$k\text{-Cov}_{2n+m}(\frac{1}{2n}, I) \sim m$-dimensional simplicial complex.

Need extension of excess 1 flows.



Theorem (Milnor, Classical Morse Theory)

Let $f : M \to \mathbb{R}$ be smooth. Let $a < b$. If $f^{-1}[a, b]$ is compact, and contains no points where $\bigtriangledown f = 0$, then $M^a = f^{-1}[-\infty, a]$ is homotopy equivalent to $M^b = f^{-1}[-\infty, b]$.

Definition (Tautological Function)

Define $\tau : Cov_n(r, Y) \to \mathbb{R}$ by $\tau(\vec{x} = (x_1, \ldots, x_n)) = \max_{y \in Y} \min_{1 \leq i \leq n} d(x_i, y)$

τ is only *piece-wise smooth*; must use techniques such as in [Agrachev, 1997].

Future Work: Non-smooth Morse Theory

Definition (Tautological Function for Double-covering)

Define $\tau : \text{2-Cov}_n(r, Y) \to \mathbb{R}$ by

$$\tau(\vec{x}) = \max_{y \in Y} \min_{1 \leq i < j \leq n} \max\{d(x_i, y), d(x_j, y)\}$$

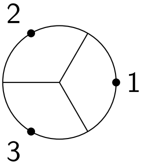
Conjecture

The only critical points of τ occur when the excess changes.

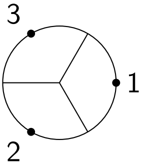
Excess 0 coverings of S^1

Space of Single Coverings of the Circle

n balls of radius $\frac{1}{2n}$, total length $n \cdot \frac{1}{n} = 1$



Permutations $123, 312, 231$ equivalent up to rotation.



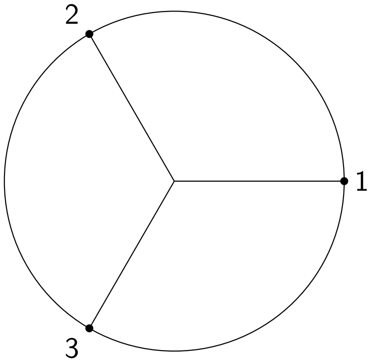
Permutations $132, 321, 213$ equivalent up to rotation.

Theorem

$$Cov_n\left(\frac{1}{2n}, S^1\right) \cong \coprod_{i=1}^{(n-1)!} S^1$$

Space of Double Coverings of the Circle, for odd n

n balls of radius $\frac{1}{n}$, total length $n \cdot \frac{1}{2n} = 2$



Theorem (same reasoning as single-covering case)

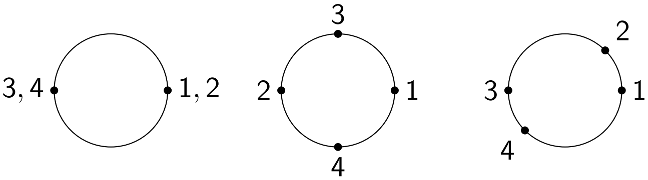
$$2\text{-}Cov_n(\tfrac{1}{n}, S^1) \cong \bigsqcup_{i=1}^{(n-1)!} S^1$$

Space of Double Coverings for $n = 2, r = \frac{1}{2}$

Both balls cover the entire circle, can be moved independently of each other.

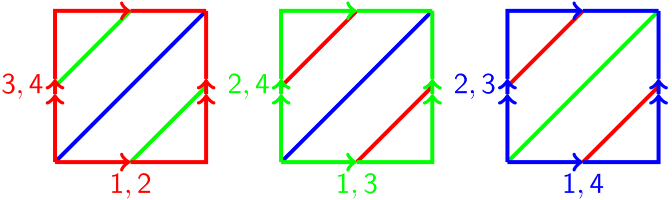
Theorem

$2\text{-}Cov_2(\frac{1}{2}, S^1) \cong S^1 \times S^1 \cong T^2$



Theorem

2-$Cov_4(\frac{1}{4}, S^1) \cong 3$ *tori, glued as below. In general, for n even,*
2-$Cov_n(\frac{1}{n}, S^1) \cong \frac{2(n-1)!}{n}$ *tori; each torus glued to* $\frac{n}{2} \cdot (2^{\frac{n-2}{2}} - 1)$ *other tori.*



$\pi_1(2 - \mathrm{Cov}_4(\frac{1}{4}, S^1)) =$
$\mathbb{Z} \times F_3$

Generalization to k-covering, Future Work

Small Result: $k\text{-Cov}_k(\frac{1}{2}, S^1) \cong T^k$.

Conjecture

$$k\text{-Cov}_n(\tfrac{k}{n}, S^1) \cong \coprod_{i=1}^{(n-1)!} S^1 \text{ if } k \nmid n.$$

Generalize to higher k, and look at higher excess.

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