Quadrature by Multipole Expansion

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Motivation

QBX works by constructing *local* expansions of layer potentials, which are functions of the form $f(x) = \int_{\partial\Omega} G(x, y)\mu(y)\, dy$. What if we decided to use *multipole* expansions instead?

- Why would we want to do this?
- What would such a scheme look like?

Motivation

Consider the case of a decaying Green's function $G(x, y)$.

- Local (polynomial) expansions do not reproduce the decay of the layer potential in the exterior domain.
- If you use multipoles ($G(x, y)$ and its derivatives) as an expansion basis, the expansion does reproduce this decay.
- Could this lead to more accurate expansions?

Derivation

We're going to work with the double layer potential in $\mathbb{R}^2$, which comes from dipoles.

Away from the curve Γ, the double layer can be shown to satisfy the complex line integral

$$D\mu(z) = -\frac{1}{2\pi} \operatorname{Im} \int_\Gamma \frac{\mu(y)}{y - z}\, dy.$$

where μ is real-valued.

Derivation

Expansions (both local and multipole) consist of source points, target points, and centers.
We're going to follow the convention:

- y = source
- z = target
- c = center

Derivation

Introduce an expansion center c into the kernel

$$\frac{1}{y-z} = \frac{1}{(y-c)-(z-c)}.$$

Assuming that $|c-z| < |c-y|$, applying the geometric series gets

$$\begin{aligned}\frac{1}{(y-c)-(z-c)} &= \frac{1}{y-c}\left(\frac{1}{1-\frac{z-c}{y-c}}\right) \\ &= \frac{1}{y-c}\left(1+\left(\frac{z-c}{y-c}\right)+\cdots\right).\end{aligned}$$

Derivation

This gives us a *Taylor series*

$$D\mu(z) = -\frac{1}{2\pi}\,\mathrm{Im}\sum_{k=0}^{\infty}\int_{\Gamma}\frac{\mu(y)(z-c)^k}{(y-c)^{k+1}}dy.$$

This is the first step to (standard) QBX.

Derivation

If we instead assume that $|c - z| > |c - y|$, the geometric series is

$$\begin{aligned}
\frac{1}{(y-c)-(z-c)} &= \frac{1}{c-z}\left(\frac{1}{\frac{y-c}{c-z}-1}\right) \\
&= \frac{1}{c-z}\left(\frac{1}{1-\frac{c-y}{c-z}}\right) \\
&= \frac{1}{c-z}\left(1+\left(\frac{c-y}{c-z}\right)+\cdots\right).
\end{aligned}$$

Derivation

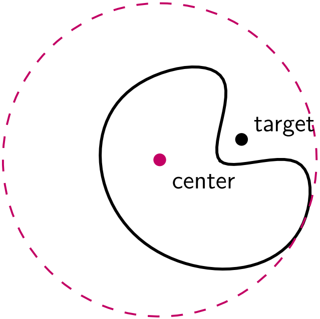
Formally, the multipole expansion of $D\mu$ takes the form:

$$D\mu(z) = -\frac{1}{2\pi} \operatorname{Im} \sum_{k=0}^{\infty} \int_{\Gamma} \mu(y) \frac{(c-y)^k}{(c-z)^{k+1}} \, dy. \tag{1}$$

This equation does not specify where to put c.

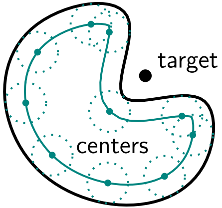
Center Placement

A valid center $c = c(t)$ *may not exist* for every target t (violates assumption $|c - y| < |c - z|$).



Center Placement

Idea is to let the center vary by source $c = c(s)$. Convergence criterion $|c(y) - y| < |c(y) - z|$ is satisfied.



FMM

Is this FMM-compatible? Yes. Insight: When discretized, centers become multipole "sources".

$$\int_{\partial\Omega} \sum_{k=0}^{p} \frac{\mu(y)(c-y)^k}{(c-z)^{k+1}}\,dy \approx \sum_{i=1}^{n}\sum_{k=0}^{p} \frac{\overbrace{w_i\mu(y_i)(c_i-y_i)^k}^{\text{source coefficient}}}{\underbrace{(c_i-z)^{k+1}}_{\text{multipole}}}$$

Results

- Error terms can be split into *truncation error* and *quadrature error*.
- We did an empirical study: How does the truncation error of QBMX compare to QBX?

Results

- We computed the truncation error in the QBMX scheme compared to the QBX scheme for a potential on the exterior of a domain. We used the double layer potential in 2 dimensions.
- We used a fixed expansion radius of $r = 0.1$. For QBX, the expansion centers were placed on the exterior of the domain, while for QBMX the centers were placed on the interior.

Results (I)

density	QBX$^{(1)}$	QBX$^{(3)}$	QBX$^{(5)}$	QBMX$^{(1)}$	QBMX$^{(3)}$	QBMX$^{(5)}$
$\sin(\tau)$	4.1(-03)	3.4(-05)	2.8(-07)	**5.2(-15)**	**5.1(-14)**	**8.1(-13)**
$\sin(3\tau)$	2.2(-02)	4.4(-04)	6.7(-06)	5.0(-03)	**6.3(-15)**	**2.7(-13)**
$\sin(5\tau)$	4.8(-02)	1.8(-03)	4.3(-05)	2.6(-02)	5.0(-05)	**5.8(-14)**

Results for unit circle

Results (II)

density	QBX$^{(1)}$	QBX$^{(3)}$	QBX$^{(5)}$	QBMX$^{(1)}$	QBMX$^{(3)}$	QBMX$^{(5)}$
$\sin(\tau)$	2.6(-03)	9.8(-05)	4.7(-06)	3.2(-03)	1.1(-05)	3.9(-07)
$\sin(3\tau)$	1.7(-02)	6.1(-04)	2.9(-05)	4.1(-03)	8.2(-05)	1.4(-06)
$\sin(5\tau)$	4.2(-02)	2.2(-03)	1.1(-04)	1.3(-02)	3.3(-04)	1.8(-06)

Results for ellipse with semiaxes $a = 2$, $b = 1$

Results (III)

density	QBX$^{(1)}$	QBX$^{(3)}$	QBX$^{(5)}$	QBMX$^{(1)}$	QBMX$^{(3)}$	QBMX$^{(5)}$
$\sin(\tau)$	5.4(-03)	7.3(-05)	1.5(-06)	2.0(-02)	1.3(-03)	9.8(-05)
$\sin(3\tau)$	4.1(-02)	1.1(-03)	2.8(-05)	5.6(-02)	4.2(-03)	3.3(-04)
$\sin(5\tau)$	1.0(-01)	5.1(-03)	1.8(-04)	1.2(-01)	1.1(-02)	1.0(-03)

Results for oval of Cassini

$$(w(\tau) = \left(\cos(2\tau) + \sqrt{a^4 - \sin^2(2\tau)}\right)^{1/2} e^{i\tau}, a = 1.15)$$

Conclusions

QBX with multipoles is possible:

- compatible with FMM
- high order (empirically)

Many open questions remain:

- In what situations is using multipoles practical?
- Can we give a good error estimate?