Quadrature by Multipole Expansion

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Motivation

QBX works by constructing local expansions of layer potentials, which are functions of the form
\[ f(x) = \int_{\partial \Omega} G(x, y) \mu(y) \, dy. \]
What if we decided to use multipole expansions instead?

- Why would we want to do this?
- What would such a scheme look like?
Consider the case of a decaying Green's function $G(x, y)$.

- Local (polynomial) expansions do not reproduce the decay of the layer potential in the exterior domain.
- If you use multipoles ($G(x, y)$ and its derivatives) as an expansion basis, the expansion does reproduce this decay.
- Could this lead to more accurate expansions?
Derivation

We’re going to work with the double layer potential in \( \mathbb{R}^2 \), which comes from dipoles. Away from the curve \( \Gamma \), the double layer can be shown to satisfy the complex line integral

\[
D\mu(z) = -\frac{1}{2\pi} \text{Im} \int_{\Gamma} \frac{\mu(y)}{y - z} \, dy.
\]

where \( \mu \) is real-valued.
Expansions (both local and multipole) consist of source points, target points, and centers. We’re going to follow the convention:

- $y = \text{source}$
- $z = \text{target}$
- $c = \text{center}$
Introduce an expansion center $c$ into the kernel

$$\frac{1}{y - z} = \frac{1}{(y - c) - (z - c)}.$$

Assuming that $|c - z| < |c - y|$, applying the geometric series gets

$$\frac{1}{(y - c) - (z - c)} = \frac{1}{y - c} \left(1 - \frac{z - c}{y - c}\right)$$

$$= \frac{1}{y - c} \left(1 + \left(\frac{z - c}{y - c}\right) + \cdots \right).$$
Derivation

This gives us a *Taylor series*

\[ D\mu(z) = -\frac{1}{2\pi} \text{Im} \sum_{k=0}^{\infty} \int_{\Gamma} \frac{\mu(y)(z - c)^k}{(y - c)^{k+1}} dy. \]

This is the first step to (standard) QBX.
Derivation

If we instead assume that \( |c - z| > |c - y| \), the geometric series is

\[
\frac{1}{(y - c) - (z - c)} = \frac{1}{c - z} \left( \frac{1}{\frac{y - c}{c - z} - 1} \right) = \frac{1}{c - z} \left( 1 - \frac{c - y}{c - z} \right) = \frac{1}{c - z} \left( 1 + \left( \frac{c - y}{c - z} \right) + \cdots \right).
\]
Formally, the multipole expansion of $D\mu$ takes the form:

$$D\mu(z) = -\frac{1}{2\pi} \text{Im} \sum_{k=0}^{\infty} \int_{\Gamma} \mu(y) \frac{(c - y)^k}{(c - z)^{k+1}} \, dy. \quad (1)$$

This equation does not specify where to put $c$. 

Derivation
Center Placement

A valid center $c = c(t)$ *may not exist* for every target $t$ (violates assumption $|c - y| < |c - z|$).
Center Placement

Idea is to let the center vary by source $c = c(s)$. Convergence criterion $|c(y) - y| < |c(y) - z|$ is satisfied.
Is this FMM-compatible? Yes. Insight: When discretized, centers become multipole “sources”.

\[ \int_{\partial \Omega} \sum_{k=0}^{p} \frac{\mu(y)(c - y)^k}{(c - z)^{k+1}} \, dy \approx \sum_{i=1}^{n} \sum_{k=0}^{p} \frac{w_i \mu(y_i)(c_i - y_i)^k}{(c_i - z)^{k+1}} \]

source coefficient

multipole
Results

- Error terms can be split into *truncation error* and *quadrature error*.
- We did an empirical study: How does the truncation error of QBMX compare to QBX?
Results

- We computed the truncation error in the QBMX scheme compared to the QBX scheme for a potential on the exterior of a domain. We used the double layer potential in 2 dimensions.

- We used a fixed expansion radius of $r = 0.1$. For QBX, the expansion centers were placed on the exterior of the domain, while for QBMX the centers were placed on the interior.
### Results (I)

<table>
<thead>
<tr>
<th>density</th>
<th>QBX(^{(1)})</th>
<th>QBX(^{(3)})</th>
<th>QBX(^{(5)})</th>
<th>QBMX(^{(1)})</th>
<th>QBMX(^{(3)})</th>
<th>QBMX(^{(5)})</th>
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</thead>
<tbody>
<tr>
<td>sin((\tau))</td>
<td>4.1(-03)</td>
<td>3.4(-05)</td>
<td>2.8(-07)</td>
<td><strong>5.2(-15)</strong></td>
<td><strong>5.1(-14)</strong></td>
<td>8.1(-13)</td>
</tr>
<tr>
<td>sin(3(\tau))</td>
<td>2.2(-02)</td>
<td>4.4(-04)</td>
<td>6.7(-06)</td>
<td>5.0(-03)</td>
<td><strong>6.3(-15)</strong></td>
<td><strong>2.7(-13)</strong></td>
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<tr>
<td>sin(5(\tau))</td>
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<td>1.8(-03)</td>
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<td>2.6(-02)</td>
<td>5.0(-05)</td>
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</tbody>
</table>

Results for unit circle
Results (II)

Results for ellipse with semiaxes $a = 2, b = 1$

<table>
<thead>
<tr>
<th></th>
<th>$\text{QBX}^{(1)}$</th>
<th>$\text{QBX}^{(3)}$</th>
<th>$\text{QBX}^{(5)}$</th>
<th>$\text{QBMX}^{(1)}$</th>
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<th>$\text{QBMX}^{(5)}$</th>
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<tbody>
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<td>1.1(-05)</td>
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<tr>
<td>$\sin(3\tau)$</td>
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<td>1.8(-06)</td>
</tr>
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Results (III)

<table>
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<tr>
<th></th>
<th>QBX(1)</th>
<th>QBX(3)</th>
<th>QBX(5)</th>
<th>QBMX(1)</th>
<th>QBMX(3)</th>
<th>QBMX(5)</th>
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</thead>
<tbody>
<tr>
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<td>7.3(-05)</td>
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<td>9.8(-05)</td>
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<tr>
<td>sin(3τ)</td>
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<td>1.1(-03)</td>
<td>2.8(-05)</td>
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<td>1.2(-01)</td>
<td>1.1(-02)</td>
<td>1.0(-03)</td>
</tr>
</tbody>
</table>

Results for oval of Cassini

\[
(w(τ) = \left( \cos(2τ) + \sqrt{a^4 - \sin^2(2τ)} \right)^{1/2} e^{iτ}, a = 1.15)
\]
Conclusions

QBX with multipoles is possible:

- compatible with FMM
- high order (empirically)

Many open questions remain:

- In what situations is using multipoles practical?
- Can we give a good error estimate?