The expected signature of a diffusion process and its PDE

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There are many computational tasks in which it is necessary to sample a given probability density (pdf). For example, one may want to evaluate $I = E[g(\eta)] = \int g(x)f(x)dx$, where $f(x)$ is a pdf and $\eta \sim f$. To perform this calculation one needs samples $X_j \sim f$. One can proceed by sampling a proposal density f_0 and approximating I as a weighted sum with weights f/f_0; for efficient computation, the weights should be close to 1, and a suitable f_0 may be hard to find, in particular when the dimension of x is very large.

Implicit sampling finds a suitable f_0 numerically. Write $F(x) = -\log f(x)$, and suppose for the moment that F is convex. Pick a reference variable ξ such that: (i) ξ is easy to sample, (ii) its pdf $g(\xi)$ has a maximum at $\xi = 0$, (iii) the logarithm of g is convex, and (iv) it is possible to write η as a function of ξ. It is often convenient to pick $\xi \sim \mathcal{N}(0, I)$; this choice does not imply any Gaussianity assumption. Then find $\phi = \min F$, and pick a sequence of independent samples ξ. For each one, solve the equation

$$F(X_j) - \phi = \frac{1}{2}\xi^T\xi, \tag{1}$$

making sure the mapping $\xi \to X$ is one-to-one and onto. The minimization of F guides the samples X_j to where the probability is high and importance sampling has been achieved.

This construction generalizes to non-convex F and can be implemented efficienty. I will carry it out in detail in a filtering problem, where it leads to a powerful new data assimilation algorithm.

(Joint work with M. Morzfeld and X. Tu).