Comparison Inequalities and Fastest-Mixing Markov Chains
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We introduce a new partial order on the class of stochastically monotone Markov kernels having a given stationary distribution \( \pi \) on a given finite partially ordered state space \( \mathcal{X} \). When \( K \preceq L \) in this partial order we say that \( K \) and \( L \) satisfy a comparison inequality. We establish that if \( K_1, \ldots, K_t \) and \( L_1, \ldots, L_t \) are reversible and \( K_s \preceq L_s \) for \( s = 1, \ldots, t \), then \( K_1 \cdots K_t \preceq L_1 \cdots L_t \). In particular, in the time-homogeneous case we have \( K^t \preceq L^t \) for every \( t \) if \( K \) and \( L \) are reversible and \( K \preceq L \), and using this we show that (for suitable common initial distributions) the Markov chain \( Y \) with kernel \( K \) mixes faster than the chain \( Z \) with kernel \( L \), in the strong sense that at every time \( t \) the discrepancy—measured by total variation distance or separation or \( L^2 \)-distance—between the law of \( Y_t \) and \( \pi \) is smaller than that between the law of \( Z_t \) and \( \pi \).

Using comparison inequalities together with specialized arguments to remove the stochastic monotonicity restriction, we answer a question of Persi Diaconis by showing that, among all symmetric birth-and-death kernels on the path \( \mathcal{X} = \{0, \ldots, n\} \), the one (we call it the uniform chain) that produces fastest convergence from initial state 0 to the uniform distribution has transition probability \( 1/2 \) in each direction along each edge of the path, with holding probability \( 1/2 \) at each endpoint.

We also use comparison inequalities

(i) to identify, when \( \pi \) is a given log-concave distribution on the path, the fastest-mixing stochastically monotone birth-and-death chain started at 0, and
(ii) to recover and extend a result of Peres and Winkler that extra updates do not delay mixing for monotone spin systems.

Among the fastest-mixing chains in (i), we show that the chain for uniform \( \pi \) is slowest in the sense of maximizing separation at every time.

(This is joint work with Jonas Kahn of University of Lille in France.)