Circumventing Schmidt's bound on discrepancy using tapered estimators

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In 1935, van der Corput raised the question of whether there could exist a sequence of numbers $x_1, x_2, \ldots$ in $[0,1)$ such that the discrepancy $D(n) = \sup_{\alpha \in [0,1)} |\#(\{1 \le i \le n : x_i < \alpha\}) - n\alpha|$ is bounded as $n \to \infty$; in 1945, Van Aardenne-Ehrenfest showed that the answer is "no", with later improvements by Roth and by Schmidt. But was van der Corput asking the right question? For many purposes, he was not.

Precisely because many of the sequences that come close to achieving the Schmidt bound (such as van der Corput's sequence $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \ldots$) have built-in periodicities, there can be better ways to use $x_1, x_2, \ldots$ to approximate the length of an interval I than $\#(\{1 \le i \le n : x_i \in I\})/n$. This will not come as a surprise to those who work in signal processing, but the insight does not seem to be present in the discrepancy theory literature.

I will describe a notion of discrepancy in which the estimator uses quadratic tapering, and I will present evidence that with this sort of tapering, one can get better estimates than are possible with the naive untapered estimator. At the very least, it appears one can trespass Schmidt's bound by a factor of $\log n$.

I will also present an easily proved but apparently new result that shows that, for one natural variant of van der Corput's question (more along the lines of the L^2 result of Roth), the correct order of magnitude of the unrescaled discrepancy is not $1/n$ but $1/n^{3/2}$.

If time permits, I will discuss the application of tapered estimation to the study of quasirandom analogues of stochastic processes that originally motivated these investigations, since this topic may be of independent interest.

Finally, I will talk about some variants of the Gauss circle problem that arise from applying the idea of tapering in the space domain rather than the time domain.