Diversions from the Koksma-Hlawka inequality

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Koksma's inequality states that if a function f has bounded variation $V(f)$ on the unit interval $[0,1)$, and $x_1, \ldots, x_N$ are N points in $[0,1)$, then

$$\left| \frac{1}{N} \sum_{j=1}^{N} f(x_j) - \int_0^1 f \right| \leq V(f) \; D_N^* \left(\{x_j\}_{j=1}^N \right) \; ,$$

where

$$D_N^* \left(\{x_j\}_{j=1}^N \right) := \sup_{0<\alpha\leq 1} \left| -\alpha + \frac{1}{N} \sum_{j=1}^{N} \chi_{[0,\alpha)}(x_j) \right| \; .$$

The term *Koksma-Hlawka inequality* refers to E. Hlawka's generalization of the above result to higher dimensions, where the intervals $[0,\alpha)$ are replaced by axis-parallel boxes anchored at the origin, and $V(f)$ becomes the variation in the sense of Hardy and Krause. Observe that many familiar functions in one variable have bounded variation, but the multi-dimensional case is more delicate, for example: the characteristic function of a polyhedron has bounded Hardy-Krause variation only if the polyhedron is an axis-parallel box.

Variations on the Koksma-Hlawka inequality have been proposed by several authors.

We describe two such variations. The first one concerns a general class of piecewise smooth functions on $\mathbb{T}^d$, the second one is especially tailored for characteristic functions of simplices.

We also consider different choices for the sequence $\{x_j\}_{j=1}^N \subset \mathbb{T}^d$.

Finally we discuss related inequalities in the general setting of metric measure spaces.

From joint papers with Luca Brandolini, William Chen, Leonardo Colzani, Giacomo Gigante