Factorization Norms and Tusnády's Problem

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Tusnády's problem in d dimensions asks for the smallest possible value $\Delta_d(n)$ for which any n-point set P in $\mathbb{R}^d$ can be two-colored so that no axis-aligned box has more than $\Delta_d(n)$ points of one color in excess of the other. In 1981, Beck showed that $\Delta_d(n)$ is bounded from below by the Lebesgue-measure discrepancy of axis-aligned boxes, which is known to be at least $\Omega(\log^{(d-1)/2} n)$. On the other hand, the best known upper bound for $\Delta_d(n)$ is considerably larger: Larsen gave a bound of $O(\log^{d+1/2} n)$, slightly improving on a previous result of Matoušek. We nearly close this gap and show a new lower bound of $\Omega(\log^{d-1} n)$. We also give a new simple proof of the upper bound. Our results are proved via the classical factorization norm γ_2, defined for a linear operator $A\colon \ell_1 \to \ell_\infty$ as the minimum norm of a factorization of A through ℓ_2. The γ_2 norm of a matrix A, taken as an operator, approximates its hereditary discrepancy up to logarithmic factors and satisfies a number of nice properties: it is efficiently computable, satisfies the triangle inequality, and is multiplicative with respect to tensor products. This makes it a powerful tool in combinatorial discrepancy theory.

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