Physical "Illustrations" of Knot Deformations and Surface Eversions

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Recently I have become interested in two problem areas where hand-drawn sketches or computer-generated illustrations were not sufficient to give me answers to some key questions. I needed to resort to more physical realizations, with which I could then play to obtain a more intuitive understanding.

The Mathematical Knot Tables give no hints of the beautiful 3D structures that some of these knots may assume, e.g., that knot 940 represents the 3-fold symmetrical "Chinese Button Knot." Since I don't know how to find the maximal possible symmetry of a given knot, I took the opposite approach: I generated knots with a specified symmetry, and then tried to find out to which knot in the Table this construction might correspond. In both these processes, knots were presented as hand-drawn sketches, CAD-models, 3D-prints, and physical realizations from pipe-cleaners or flexible hoses.

The second problem domain concerns the eversion of 2-manifolds with embedded knots on them. Many good movies exist that show how to turn a sphere or torus inside-out via a regular homotopy. However, there are other ways to "invert" a surface. The simplest one is a geometrical reflection, e.g., by inverting the x-values of all vertices supporting a faceted mesh description of a given 2-manifold. This process would convert a right-twisting (3,2)-Torus knot embedded on a torus into a mirrored, left-twisting (3,-2)-Torus knot. Yet another way to "invert" a surface is based on a more "physical" process, where one or more punctures are introduced, and the surface is then pulled through these punctures to bring its inside to the outside. In this process the above (3,2)-Torus knot will be converted into a (2,3)-Torus knot. The question arises: Is this "physical" inversion always possible for handle-bodies of arbitrary complexity? – and: How many punctures would be needed?
Other questions that arise in this context are: What is the result on the above torus when using Cheritat's torus eversion process? – Will Chéritat's torus eversion process also work for handle-bodies of higher genus? How do we use illustrations and visualization tools to convey these facts most easily to non-mathematicians? In my own attempts of finding answers to some of these questions, I have also used surfaces made of a collection of paper strips or glued-together plastic bags, with knots or graphs drawn upon them.