Of Kato and Prandtl

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We consider the behavior of a classical low-viscosity (high Reynolds number) incompressible fluid in the presence of a boundary, using classical no-slip boundary conditions. Our focus is on generic initial data and geometries---that is, without special symmetries or properties such as analyticity.

Ludwig Prandtl in 1904 introduced what are now called the Prandtl equations, heuristically derived equations hypothesized to describe laminar flow in a thin boundary layer. Prandtl paints a very appealing physically motivated picture of the boundary behavior of a low-viscosity fluid, but the resulting asymptotic expansion has never been proved mathematically to hold (or to fail).

Kato, by contrast, in his seminal 1983 paper, sought not to describe in any detail the behavior of the fluid near the boundary, but simply to ask whether as the viscosity is theoretically taken to zero the viscous solutions, as described by the Navier-Stokes equations, converge in the energy norm to the solution of the inviscid fluid equations, the Euler equations. This type of convergence is called the (classical) vanishing viscosity limit. Kato found simple necessary and sufficient conditions on the solution to the Navier-Stokes equations to guarantee such convergence.

These two ways of looking at low-viscosity fluids seem highly related, yet relationships between them have not been firmly established. There are a number of clear reasons for this: First, the Prandtl theory intimately involves a boundary layer of width proportional to the square root of the viscosity. But Kato's theory is concerned solely with a width directly proportional to the viscosity. Second, Prandtl's equations have been shown to be well-posed only in very special circumstances and shown to be ill-posed in certain spaces (in the sense of instability); nor is there a universally adopted agreement on how (in what norms, for instance) the Prandtl solutions are to describe the Navier-Stokes solutions. Third, Kato's theory only gives necessary and sufficient conditions on the solutions to obtain convergence---whether such conditions hold in general or fail in any one circumstance, is a wide open problem.

In this talk we begin an integration of the approaches to low viscosity fluids of Prandtl and Kato. We will give an overview of the state of the art of the Kato theory then show how the Kato theory constrains the best one can hope for from the Prandtl theory.