The three-wave PDEs for resonantly interacting triads

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The "3-wave equations" describe the simplest possible nonlinear interaction among dispersive waves in a medium with no dissipation. If each set of waves is spatially uniform, then the 3-wave equations appear as ordinary differential equations (ODEs); if the wave amplitudes vary in space, then the equations appear as partial differential equations (PDEs). In either case, the equations are nonlinear, with complex-valued solutions. This family of models describes the evolution of capillary-gravity waves on the ocean's free surface, of internal waves in the ocean's interior, of electromagnetic waves in nonlinear optics, of Langmuir waves in a plasma, and elsewhere.

In addition to their physical significance, the equations have interesting mathematical structure. The (nonlinear) ODEs are completely integrable in the sense of Liouville: they are solved in terms of "action-angle" variables, and the general solution can be expressed in terms of elliptic functions (see Bretherton, 1964).

The (nonlinear) PDEs are also completely integrable (in the sense of soliton theory): Zakharov & Manakov (1973, 1976) and Kaup (1976) used inverse scattering theory to solve the initial-value problem for these PDEs in unbounded space. Somewhat surprisingly, the same problem with periodic (or other) boundary conditions on a finite interval remains open.

The objective of the work presented here is to construct the "general solution" of the PDEs, which would be valid for a wide variety of boundary conditions, and with no restriction to initial-value problems. The 3-wave ODEs can be written as six real-valued ODEs, and its general solution consists of six real-valued functions that solve the ODEs, and that contain six, independent constants of the motion. Along the same lines, the general solution of the PDEs is obtained by replacing the six constants for the ODEs with six, independent real-valued functions of space for the PDEs. PDEs for which a general solution is known are rare: the only nontrivial example known to the speaker is D'Alembert's solution of the one-dimensional (linear) wave equation.

This work is still in progress, but enough of the structure is in place to describe the general procedure. The current work builds on earlier work by Ruth Martin (PhD thesis, 2015) and Martin & Segur (2016).