Almost orthogonal vectors

William J. Martin, Worcester Polytechnic Institute

Let $N(d)$ denote the maximum size of a set of lines through the origin in $\mathbb{R}^d$ pairwise at angle 89° or more. Since $N(d) = d$ for $d \leq 57$, some are surprised to learn that $N(d)$ is an exponential function of d, as proved by Shannon long ago. This motivates us to examine more carefully spherical codes with all inner products within ε of zero.

Let $\varepsilon = \varepsilon(d)$ be a positive decreasing function of d tending toward zero. We ask for the largest size of a set X of unit vectors in $\mathbb{R}^d$ such that $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \varepsilon(d)$ for all $\mathbf{x}, \mathbf{y} \in X$ with $\mathbf{x} \neq \mathbf{y}$. How large must $\varepsilon(d)$ be in order to allow $|X|$ to grow exponentially with d? Where does linear growth give way to quadratic growth? I have mostly questions and few answers.

In this talk, we will explore bounds and constructions for spherical codes with all inner products very close to zero. I will discuss connections to frames, association schemes and quantum information theory and I will mention new results of Bukh and Cox on the optimal value of ε in the case where $|X| = d + k$ where k is constant.