# Study group proposal: semistable reduction

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Given a family of varieties $X \to B$, to what extent can we resolve its singularities? Let's focus on characteristic 0.

- Hironaka's theorem solves the case $\dim B = 0$: resolve $X$.
- The book [6] answers the question when $\dim B = 1$, making the family semistable, and
- on Page VII Mumford suggests generally pursuing the higher dimensional base case.
- In [1, Conjecture 0.5] Karu and I state the general semistable reduction problem precisely, and prove a weaker version [1, Theorem 0.3].
- The conjecture was settled in fiber dimension $\leq 3$ by Karu in [5]. Finally
- The general case was only recently settled by Adiprasito, Liu and Temkin [3], using ideas coming from [4].

I propose to lead a reading seminar working through the highlights of these works, and leading to a remaining open question, which I believe achievable in the duration of this program.

0.1. Background: *Log smooth reduction.* In [1] Karu and I proved that any morphism of varieties in characteristic 0 can be made logarithmicelly smooth.

**Theorem 0.1.1** ([1], Theorem 2.1)**.** *Let $X \to B$ be a projective dominant morphism of varieties in characteristic 0. There is a modification $X' \to B'$ which is logarithmically smooth.*

In [2] Temkin, Włodarczyk and I prove that this can be done in a relatively functorial manner. We are working to upgrade the functoriality statement, in particular compatibility with arbitrary group actions in this regime.

0.2. Background: *semistable reduction.* Our paper [1] also proves *weak semistable reduction*, which using logarithmic geometry reads as follows:

**Theorem 0.2.1** ([1], Theorem 0.3)**.** *For a log smooth $X \to B$ there is an alteration $B_1 \to B$ and a modification $X_1 \to X \times_B^{fs} B_1$ of the saturated pullback such that $X_1 \to B_1$ is log smooth and saturated, namely toroidal, flat, with reduced fibers.*

In [1, Conjecture 0.5] Karu and I conjectured that the morphism can be made *semistable*. This means that locally $X_1 \to B_1$ is of the form

$$\begin{array}{rcl} t_1 & = & y_1 \cdots y_{k_1} \\ \vdots & & \vdots \\ t_\ell & = & y_{k_{\ell-1}+1} \cdots y_{k_\ell}, \end{array}$$

in other words it is, locally, a product of $\ell$ one-parameter semistable families.

[1, Conjecture 8.4] is a purely combinatorial conjecture which is shown in [1, Proposition 8.5] to be equivalent to [1, Conjecture 0.5].

Finally, in [3], Adiprasito Liu and Temkin prove these conjectures:

**Theorem 0.2.2** ([3]). *For a log smooth $X \to B$ there is an alteration $B_1 \to B$ and a modification $X_1 \to X \times_B^{fs} B_1$ of the saturated pullback such that $X_1 \to B_1$ is semistable. In particular* [1, Conjecture 8.4] *and* [1, Conjecture 0.5] *hold true.*

This in particular answers in most precise manner Mumford wish from [6, Page VII].

0.3. Remaining challenge: *Functorial result.* Semistability is the best type of singularities one can construct for families. But it is inherently not stable under base change, thus not "permanent" in Raynaud's sense. I have a concrete conjecture to overcome this, which I believe can be resolved, perhaps easily, during the program, once we understand the methods and results of [3].

## References


[1] D. Abramovich and K. Karu. Weak semistable reduction in characteristic 0. *Invent. Math.*, 139(2):241–273, 2000.

[2] Dan Abramovich, Michael Temkin, and Jarosław Włodarczyk. Relative desingularization and principalization of ideals. *arXiv e-prints*, page arXiv:2003.03659, March 2020.

[3] K. Adiprasito, G. Liu, and M. Temkin. Semistable reduction in characteristic 0, November 2018. http://arxiv.org/abs/1810.03131.

[4] Christian Haase, Andreas Paffenholz, Lindsay C. Piechnik, and Francisco Santos. Existence of unimodular triangulations - positive results. *arXiv e-prints*, page arXiv:1405.1687, May 2014.

[5] K. Karu. Semistable reduction in characteristic zero for families of surfaces and threefolds. *Discrete Comput. Geom.*, 23(1):111–120, 2000.

[6] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.