On metric embeddings, shortest path decompositions and face cover of planar graphs

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Shortest path decomposition (SPD) is a hierarchical partition of a graph using shortest paths. Every (weighted) path graph has an SPD of depth 1. A graph G has an SPD of depth k if after removing some shortest path P, every connected component in $G\setminus P$ has an SPD of depth $k-1$. We present a novel metric embedding technique. Our main result is that any graph with SPD of depth k, can be embedded into ℓ_p with distortion $O(k^{\min\{\frac12,\frac1p\}})$.

Every pathwidth k graph has an SPD of depth k, thus an $O(\sqrt{k})$ distortion embedding into ℓ_1 follows. This is a super-exponential improvement over the best previous bound of Lee and Sidiropoulos [Combinatorica13]. For $p>2$ our embedding also implies improved distortion on bounded treewidth k graphs ($O((k\log n)^{\frac{1}{p}})$). For asymptotically large p, our results also implies improved distortion on graphs excluding a minor.

Consider a planar graph with a terminal set K that can be covered by γ faces. The goal is to embed the terminal set K into ℓ_1. In a recent paper Krauthgamer, Lee and Rika [SODA19] showed an upper bound of $O(\log \gamma)$ on the distortion, improving previous results by Lee and Sidiropoulos [STOC09] and Chekuri et. al. [J.Comb.Theory13].
Based on SPD and truncated embeddings, we prove an upper bound of $O(\sqrt{\log\gamma})$ on the distortion.

It is well known that the flow-cut gap equals to the distortion of the best embedding into ℓ_1. In particular, our result provides a polynomial time $O(\sqrt{k})$-approximation to the sparsest cut problem for pathwidth k graphs, and in addition an $O(\sqrt{\log \gamma})$-approximation for planar graph, for the case where all the demand pairs can be covered by γ faces.