Dichotomy Theorems for Counting Problems

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There has been remarkable progress in the classification program of the complexity of counting problems. This program is carried out in at least three interrelated formulations: Graph Homomorphisms, Counting CSP, and Holant Problems which are inspired by Holographic Algorithms of Valiant. In each formulation, complexity dichotomy theorems have been achieved which classify {\it every} problem in a given class to be either solvable in polynomial time or \#P-hard.

This talk will focus on Graph Homomorphism. It was defined by Lov\'{a}sz (1967) and has been studied intensively over the decades. It is also called the Partition Function:

Given an $m \times m$ symmetric matrix A over the complex field, compute the Partition Function $Z_A(\cdot)$, where for an arbitrary input graph G, \[Z_A(G) = \sum_{\xi:V(G)\rightarrow [m]} \prod_{(u,v)\in E(G)} A_{\xi(u),\xi(v)}.\]

In this general setting, it encompasses many counting problems such as counting vertex covers, independent sets, graph colorings etc. We prove a complexity dichotomy theorem in this most general setting, that the problem $Z_A(\cdot)$ is either computable in P or \#P-hard, with an explicit decision criterion on A.
The complex field affords much possibility for cancelations (think of the permanent versus determinant.) Group theoretic properties and character sums play a major role. In the complex domain, there are also natural connections to Holographic Algorithms. Joint work with Xi Chen of Columbia and Pinyan Lu of MSRA. Paper available on http://pages.cs.wisc.edu/~jyc/