Geometry of Orbits of Permanents and Determinants

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Let $\mathfrak{v}$ be a complex vector space of dimension m and let $E := \mathfrak{v} \otimes \mathfrak{v}^* = \operatorname{End} \mathfrak{v}$. Consider $\det \in Q := S^m(E^*)$, where det is the function taking determinant of any $X \in \operatorname{End} \mathfrak{v}$. Fix a basis $\{e_1, \dots, e_m\}$ of $\mathfrak{v}$ and a positive integer $n < m$ and consider the function $\mathsf{p} \in Q$, defined by $\mathsf{p}(X) = x_{1,1}^{m-n} \operatorname{perm}(X^o)$, X^o being the component of X in the right down $n \times n$ corner, where any element of $\operatorname{End} \mathfrak{v}$ is represented by a $m \times m$-matrix $X = (x_{i,j})_{1 \leq i,j, \leq m}$ in the basis $\{e_i\}$ and perm denotes the permanent. The group $G = \operatorname{GL}(E)$ canonically acts on Q. Let $\mathcal{X}_{\det}$ (resp. $\mathcal{X}_{\mathsf{p}}$) be the G-orbit closure of det (resp. p) inside Q. Then, $\mathcal{X}_{\det}$ and $\mathcal{X}_{\mathsf{p}}$ are closed (affine) subvarieties of Q which are stable under the standard homothecy action of $\mathbb{C}^*$ on Q. Thus, their affine coordinate rings $\mathbb{C}[\mathcal{X}_{\det}]$ and $\mathbb{C}[\mathcal{X}_{\mathsf{p}}]$ are nonnegatively graded G-algebras over the complex numbers $\mathbb{C}$.

The aim of this talk is to study some geometric results about the varieties $\mathcal{X}_{\det}$ and $\mathcal{X}_{\mathsf{p}}$ and to study $\mathbb{C}[\mathcal{X}_{\det}]$ and $\mathbb{C}[\mathcal{X}_{\mathsf{p}}]$ as G-modules. The work is motivated by the geometric approach initiated by Mulmuley-Sohoni to solve the Valiant's conjecture in Geometric Complexity Theory.