A Robust Methodology for the Numerical Solution of Fully Nonlinear Elliptic Problems

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The main goal of this lecture is to briefly discuss some nonlinear least squares methods for the numerical solution of some fully nonlinear second order elliptic problems, such as the following Dirichlet problem for the Monge-Ampère equation:

(MA) $$\det \mathbf{D}^2 u = f\,(>0) \text{ in } \Omega,\ u = g \text{ on } \partial\Omega,$$

with $\Omega \subset \mathbf{R}^d, d \geq 2$.

Assuming that the data are smooth enough to be compatible with the existence of solutions in $H^2(\Omega)$, the least-squares based methodology we employ will solve the nonlinear elliptic equations under consideration as nonlinear bi-harmonic problems giving us access to well documented solution methods. Albeit intended to look for solutions with the $H^2(\Omega)$-regularity, the methodology we will discuss is robust enough, as shown by numerical experiments, to

(i) Compute solutions even if the data lack the regularity necessary to have solutions in $H^2(\Omega)$.

(ii) Compute generalized solutions if the original problem as no smooth solutions despite the smoothness of its data (this is the case if, for example, we have in (MA): $\Omega = (0, 1)^2, f = 1$ and $g = 0$).

Convincing examples of robustness will be given via the results of numerical experiments. They concern (MA) and the following Pucci's boundary value problem

$$\begin{cases} \alpha |\nabla^2 u|^2 + (\alpha - 1)^2 \det \mathbf{D}^2 u = 0 \text{ in } \Omega, \\ \nabla^2 u \leq 0 \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega, \end{cases}$$

with $\alpha > 1$.