Spectral discretization and fast solvers for fractional PDEs

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Existing numerical methods for fractional PDEs suffer from low accuracy and inefficiency in 3-D problems or in long-time integrations.Here, we develop a spectrally accurate unified Petrov-Galerkin (PG) spectral method for the general Fractional Partial Differential Equations (FPDEs) of the form ${}_0\mathcal{D}_t^{\tau} u + \sum_{j=1}^{d} c_{x_j} [\, {}_{a_j}\mathcal{D}_{x_j}^{\sigma_j} u] - \gamma u = f$, τ, $\sigma_j \in (0,2)$, in a $(1+d)$-dimensional *time-space* domain subject to Dirichlet initial and boundary conditions. The unified PG spectral method applies to the whole family of linear *hyperbolic*-, *parabolic*- and *elliptic*-like equations. We develop our PG method based on a new spectral theory for fractional Sturm-Liouville problems (FSLPs), recently introduced in [1]. Specifically, we employ the eigenfunctions of the FSLP of *first* kind (FSLP-I), called *Jacobi polyfractonomial*s, as temporal/spatial bases. Next, we construct a different space for test functions from polyfractonomial eigenfunctions of the FSLP of *second* kind (FSLP-II). Besides the high-order spatial dicretization in our PG methods, we demonstrate their efficiency and spectral accuracy in time-integration schemes for solving time-dependent FPDEs as well, rather than employing algebraically accurate traditional methods, especially when $\tau = 1$. Moreover, we formulate a general fast linear solver based on the eigen-pairs of the corresponding temporal and spatial mass matrices with respect to the stiffness matrices, which significantly reduces the computational cost. We exhibit that this framework can reduce to *hyperbolic* FPDEs such as time- and space-fractional advection (TSFA), *parabolic* FPDEs such as time- and space-fractional diffusion (TSFD) model, and *elliptic* FPDEs such as fractional Helmholtz/Poisson equations with the same ease and cost. Several numerical tests confirm the efficiency and spectral convergence of the unified PG spectral method for the aforementioned families of FPDEs. Moreover, we demonstrate the efficiency of the new approach in higher-dimensions e.g., (1+3), (1+5) and (1+9)-dimensional problems.

References

[1] M. Zayernouri, G. E. Karniadakis, Fractional Sturm-Liouville eigen-problems: Theory and numerical approximations, J. Comp. Physics 47-3 (2013) 2108–2131.

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