# The Grunbaum-Hadwiger-Ramos hyperplane mass partition problem

Pavle Blagojevic, Freie Universität Berlin

In 1960 Branko Grünbaum suggested the following innocent-looking problem:

**The Grünbaum hyperplane mass partition problem.** *Can any convex body in $\mathbb{R}^d$ be cut into $2^d$ pieces of equal volume by $d$ suitably-chosen affine hyperplanes?*

As Grünbaum noted, this is quite easy to prove for $d \leq 2$. In 1966 Hadwiger answered Grünbaum's question (positively) for $d = 3$, while solving a problem raised by J. W. Jaworowski (Oberwolfach, 1963). Grünbaum's question was independently raised in Computational Geometry, motivated by the construction of data structures for range queries.
In 1984 Avis answered Grünbaum's problem negatively for $d \geq 5$. Indeed, one cannot expect a positive answer there, since $d$ hyperplanes in $\mathbb{R}^d$ can be described by $d^2$ parameters, while the hyperplanes one is looking for need to satisfy $2^d - 1$ independent conditions, and $2^d - 1 > d^2$ for $d > 4$. The case $d = 4$ was and still is an open problem.

In 1996 Ramos formulated the following general version of the hyperplane mass partition problem for several masses:

**The Grünbaum–Hadwiger–Ramos problem.** *For each $j \geq 1$ and $k \geq 1$, determine the smallest dimension $d = \Delta(j, k)$ such that for every collection $\mathcal{M}$ of $j$ masses on $\mathbb{R}^d$ there are $k$ affine hyperplanes that cut each of the $j$ masses into $2^k$ equal pieces.*

The special case $\Delta(j, 1) = j$ of the Grünbaum–Hadwiger–Ramos problem, for a single hyperplane ($k = 1$), is settled by the ham-sandwich theorem, which was conjectured by Steinhaus and proved by Banach in 1938. The following lower bound for the function $\Delta(j, k)$ was derived by Avis (for $j = 1$) and Ramos, while the upper bound was obtained by Mani-Levitska, Vrećica & Živaljević:

$$\left\lceil \frac{2^k - 1}{k} j \right\rceil \leq \Delta(j, k) \leq j + (2^{k-1} - 1)2^{\lfloor \log_2 j \rfloor}.$$

Here $2^{\lfloor \log_2 j \rfloor}$ is $j$ rounded down to the nearest power of 2, so $\frac{1}{2}j < 2^{\lfloor \log_2 j \rfloor} \leq j$.

In addition to the general lower and upper bounds, a number of papers have treated special cases, reductions, and relatives of the problem. It was recently documented that, however, quite a number of published proofs do not hold up upon critical inspection, and indeed some of the approaches employed cannot work.

In this talk, using the relative equivariant obstruction theory in combination with the "join configuration space / test map scheme" and the study of Gray codes we prove that:

**Theorem.** *For $t \geq 1$ the following instances of the Ramos conjecture hold:*

(1) $\Delta(2^t - 1, 2) = 3 \cdot 2^{t-1} - 1$,
(2) $\Delta(2^t, 2) = 3 \cdot 2^{t-1}$,
(3) $\Delta(2^t + 1, 2) = 3 \cdot 2^{t-1} + 2$,
(4) $\Delta(2, 3) = 5$,
(5) $\Delta(4, 3) = 10$.

Consequently, $4 \leq \Delta(1, 4) \leq 5$ and $8 \leq \Delta(2, 4) \leq 10$.

(This is a joint work with Florian Frick, Albert Haase, and Günter M. Ziegler)