Back-stable Schubert calculus

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Type A Dynkin diagram with vertex set $\mathbb{Z}$:

$$\cdots \;\text{———}\; -2 \;\text{———}\; -1 \;\text{———}\; 0 \;\text{———}\; 1 \;\text{———}\; 2 \;\text{———}\; \cdots$$

Fl: infinite flag ind-variety

Back-stable Schubert calculus: study of $H^*(\mathrm{Fl})$

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Back-stable Schubert calculus: study of $H^*(\mathrm{Fl})$

Apply to affine flag ind-varieties and other settings:

algebra		module
nilHecke	$H_*^G(\mathrm{Fl} \times \mathrm{Fl})$	$H_T^*(\mathrm{Fl})$
K-nilHecke	$K_*^G(\mathrm{Fl} \times \mathrm{Fl})$	$K_T^*(\mathrm{Fl})$
affine Hecke	$K_*^{G\times\mathbb{C}^*}(T^*(\mathrm{Fl}) \times_{\mathcal{N}} T^*(\mathrm{Fl}))$	$K_{T\times\mathbb{C}^*}^*(\mathrm{Fl})$

[Su, Zhao, Zhong] [Aluffi, Mihalcea, Schuermann, Su] stable bases

Infinite flags Fl and Infinite Grassmannian Gr

$$\mathbb{C}^{[-\infty,\infty)} = \{(\cdots, c_{-1}, c_0, c_1, \dots) \mid c_i \in \mathbb{C},\ c_i = 0 \text{ for } i \gg 0\}$$

$$E_k = \prod_{i \le k} \mathbb{C} e_i$$

standard flag $$E_\bullet = (\cdots \subset E_{-1} \subset E_0 \subset E_1 \subset \cdots)$$

Fl: set of flags $F_\bullet = (F_i \mid i \in \mathbb{Z})$ such that

- $F_i = E_i$ for $i \ll 0$ and $i \gg 0$
- $\dim F_i / F_{i-1} = 1$ for all $i \in \mathbb{Z}$

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- $F_a \subset V \subset F_b$ for some $a \le 0 < b$
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$\mathrm{Fl}_+ \subset \mathrm{Fl}$: $F_i \neq E_i$ only for $i > 0$
$\mathrm{Fl}_- \subset \mathrm{Fl}$: $F_i \neq E_i$ only for $i < 0$

Fibration

$$
\begin{aligned}
\mathrm{Fl}_- \times \mathrm{Fl}_+ &\to \mathrm{Fl} \to \mathrm{Gr} \\
F_\bullet &\mapsto F_0 \\
H^*(\mathrm{Fl}) &\cong H^*(\mathrm{Gr}) \otimes H^*(\mathrm{Fl}_-) \otimes H^*(\mathrm{Fl}_+) \\
&\cong \Lambda(x_-) \otimes \mathbb{Q}[x_-] \otimes \mathbb{Q}[x_+] \\
&\cong \Lambda(x_-) \otimes \mathbb{Q}[x]
\end{aligned}
$$

algebra isomorphism!

$x_- = (\dots, x_{-2}, x_{-1}, x_0)$
$x_+ = (x_1, x_2, \dots)$
$x = x_- \cup x_+$
$\Lambda(x_-)$: symmetric functions

Find Schubert basis by back-stable limit

Familiar example of back-stable limit: Schur functions

Basis of cohomology: Schur polys indexed by min. coset reps. in $S_n/(S_k \times S_{n-k})$ or partitions $\lambda \subset k \times (n-k)$ rectangle

$$H^*(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \subset k \times (n-k)} \mathbb{Q} s_\lambda(x_{1-k}, \dots, x_0)$$

$$s_1, s_2, \dots, s_{k-1}, s_k, s_{k+1}, \dots, s_{n-1}$$

$$s_{1-k}, \dots, s_{-1}, s_0, s_1, \dots, s_{n-k-1}$$

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Omitted reflection has been shifted from k back to 0.

Take limit:

$$\mathrm{Gr} = \bigcup_{\substack{k \to \infty \\ n-k \to \infty}} \mathrm{Gr}(k,n)$$

$$H^*(\mathrm{Gr}) \cong \Lambda(x_-) = \bigoplus_{\lambda \in \mathbb{Y}} \mathbb{Q} s_\lambda(x_-)$$

$\mathbb{Y}$: set of partitions

Permutations

$s_i = (i, i+1)$

$S_{\mathbb{Z}} = \langle \dots, s_{-1}, s_0, s_1, \dots \rangle$	
$S_+ = \langle s_1, s_2, \dots \rangle$	permutes $\mathbb{Z}_{>0}$
$S_- = \langle \dots, s_{-2}, s_{-1} \rangle$	permutes $\mathbb{Z}_{\leq 0}$
$S_{\neq 0} = S_- \times S_+$	no s_0
$S^0_{\mathbb{Z}} = \{w \in S_{\mathbb{Z}} \mid ws_i > w \text{ for all } i \neq 0\}$	Grass. perms
$\mathbb{Y} =$ partitions	
$\mathbb{Y} \xrightarrow{\cong} S^0_{\mathbb{Z}}$	$\lambda \mapsto w_\lambda$
$w_{\lambda/\mu} = w_\lambda w_\mu^{-1}$	$\mu \subset \lambda$

Divided difference operators ∂_i on $\mathbb{Q}[x]$

$$\partial_i(f) = \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots,)}{x_i - x_{i+1}}$$

Lemma

- *image*(∂_i) = *ker*$(\partial_i) = s_i$*-invariants. In particular* $\partial_i^2 = 0$.
- $\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$ *and* $\partial_i \partial_j = \partial_j \partial_i$ *for* $|i - j| \geq 2$.

Schubert polynomials [Lascoux, Schützenberger]

Theorem

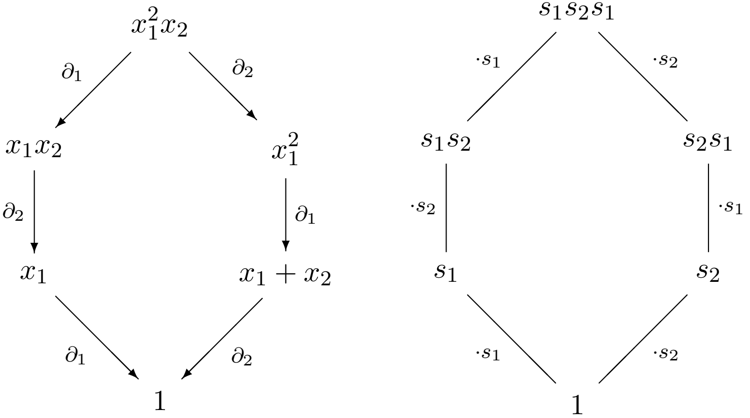
There is a unique family $\{\mathfrak{S}_w \mid w \in S_+\} \subset \mathbb{Q}[x_+]$ *such that*

- $\mathfrak{S}_{\mathrm{id}} = 1$
- $\mathfrak{S}_w$ *is homogeneous of degree* $\ell(w)$
- $$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } ws_i < w \\ 0 & \text{otherwise.} \end{cases}$$

Moreover

- $\{\mathfrak{S}_w \mid w \in S_+\}$ *is a* $\mathbb{Q}$*-basis of* $\mathbb{Q}[x_+] \cong H^*(\mathrm{Fl}_+)$.
- $\mathfrak{S}_w$ *is* s_i*-symmetric if* $w(i) < w(i+1)$.
- $$\mathfrak{S}_{w_0^{(n)}} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \qquad w_0^{(n)} \in S_n \text{ longest element}$$

Examples



$$\mathfrak{S}_{s_i} = x_1 + x_2 + \cdots + x_i.$$

Back-stable Schubert polynomials [Knutson]

$\text{shift}(s_i) = s_{i+1} \qquad \text{shift}(x_k) = x_{k+1}$

Definition

For $w \in S_\mathbb{Z}$ the back-stable Schubert polynomial $\overleftarrow{\mathfrak{S}}_w$ is

$$\overleftarrow{\mathfrak{S}}_w = \lim_{p \to -\infty} \text{shift}^{p-1}(\mathfrak{S}_{\text{shift}^{1-p}(w)})$$

Monomial expansion is well-defined (by e. g. Billey-Jockusch-Stanley formula)

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$$\overleftarrow{\mathfrak{S}}_{\mathrm{shift}(w)} = \mathrm{shift}(\overleftarrow{\mathfrak{S}}_w)$$

Examples

$$\overleftarrow{\mathfrak{S}}_{s_3} = \cdots + x_{-1} + x_0 + x_1 + x_2 + x_3$$
$$\overleftarrow{\mathfrak{S}}_{s_{-2}} = \cdots + x_{-3} + x_{-2}$$

Theorem (Lam, SJ Lee, S.)

$\{\overleftarrow{\mathfrak{S}}_w \mid w \in S_{\mathbb{Z}}\}$ is the unique family of elements of $\overleftarrow{R} = \Lambda(x_-) \otimes \mathbb{Q}[x]$ such that

1. $\overleftarrow{\mathfrak{S}}_{\mathrm{id}} = 1$
2. $\overleftarrow{\mathfrak{S}}_w$ *is homogeneous of degree* $\ell(w)$
3. $$\partial_i \overleftarrow{\mathfrak{S}}_w = \begin{cases} \overleftarrow{\mathfrak{S}}_{ws_i} & \textit{if } ws_i < w \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover they form a $\mathbb{Q}$-basis of $\overleftarrow{R} \cong H^*(\mathrm{Fl})$.

variety	Dynkin nodes	ring	basis	indexing set
Fl	$\mathbb{Z}$	$\Lambda(x_-) \otimes \mathbb{Q}[x]$	$\overleftarrow{\mathfrak{S}}_w$	S_∞
Gr	$\mathbb{Z}$ "mod" $(\mathbb{Z} - 0)$	$\Lambda(x_-)$	$s_\lambda(x_-)$	$\mathbb{Y} \cong S^0_{\mathbb{Z}}$
Fl_+	$\mathbb{Z}_{>0}$	$\mathbb{Q}[x_+]$	$\mathfrak{S}_w$	S_+
Fl_-	$\mathbb{Z}_{<0}$	$\mathbb{Q}[x_-]$	???	S_-

“Negative” Schubert polynomials

Definition

$$S_{\mathbb{Z}} \xrightarrow{\text{rev}} S_{\mathbb{Z}} \qquad \mathbb{Q}[x] \xrightarrow{\text{rev}} \mathbb{Q}[x]$$

$$\operatorname{rev}(s_i) = s_{-i} \qquad \text{for all } i \in \mathbb{Z}$$

$$\operatorname{rev}(x_k) = -x_{1-k} \qquad \text{for all } k \in \mathbb{Z}$$

$$\alpha_i = x_i - x_{i+1}$$

$$\operatorname{rev}(\alpha_i) = \alpha_{-i} \qquad \text{for all } i \in \mathbb{Z}$$

For $w \in S_-$ the “negative” Schubert polynomial $\mathfrak{S}_w \in \mathbb{Q}[x_-]$ is

$$\mathfrak{S}_w = \operatorname{rev}(\mathfrak{S}_{\operatorname{rev}(w)})$$

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$$\mathfrak{S}_{s_{-2}} = \operatorname{rev}(\mathfrak{S}_{s_2}) = \operatorname{rev}(x_1 + x_2) = -x_0 - x_{-1}$$

$$\mathfrak{S}_{s_{-3}s_{-2}s_{-1}} = \operatorname{rev}(\mathfrak{S}_{s_3 s_2 s_1}) = \operatorname{rev}(x_1^3) = -x_0^3$$

Basis

Apply rev to

$$H^*(\mathrm{Fl}_+) \cong \mathbb{Q}[x_+] = \bigoplus_{w \in S_+} \mathbb{Q}\mathfrak{S}_w$$

Get

$$H^*(\mathrm{Fl}_-) \cong \mathbb{Q}[x_-] = \bigoplus_{w \in S_-} \mathbb{Q}\mathfrak{S}_w$$

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Combine:

For $w = w_- w_+ \in S_{\neq 0} = S_- \times S_+$ define

$$\mathfrak{S}_w = \mathfrak{S}_{w_-}\mathfrak{S}_{w_+} \in \mathbb{Q}[x_-] \otimes \mathbb{Q}[x_+] \cong \mathbb{Q}[x]$$

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Basis:

$$H^*(\mathrm{Fl}_- \times \mathrm{Fl}_+) \cong \mathbb{Q}[x] = \bigoplus_{w \in S_{\neq 0}} \mathbb{Q}\mathfrak{S}_w$$

Two bases for $\overleftarrow{R} = \Lambda(x_-) \otimes \mathbb{Q}[x]$

First basis: Back stable Schubs

$$H^*(\mathrm{Fl}) \cong \overleftarrow{R} = \Lambda(x_-) \otimes \mathbb{Q}[x] = \bigoplus_{w \in S_\mathbb{Z}} \mathbb{Q}\overleftarrow{\mathfrak{S}}_w$$

$$H^*(\mathrm{Gr}) \cong \Lambda(x_-) = \bigoplus_{\lambda \in \mathbb{Y}} \mathbb{Q}s_\lambda(x_-)$$

$$H^*(\mathrm{Fl}_- \times \mathrm{Fl}_+) \cong \mathbb{Q}[x] = \bigoplus_{v \in S_{\neq 0}} \mathbb{Q}\mathfrak{S}_v$$

$$\overleftarrow{R} = \bigoplus_{(\lambda, v) \in \mathbb{Y} \times S_{\neq 0}} \mathbb{Q}s_\lambda(x_-)\mathfrak{S}_v$$

Second basis: Schur functions times Schubs.

Change of basis coefficients are ...

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$$H^*(\mathrm{Fl}_- \times \mathrm{Fl}_+) \cong \mathbb{Q}[x] = \bigoplus_{v \in S_{\neq 0}} \mathbb{Q} \mathfrak{S}_v$$

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Change of basis coefficients are ...

Edelman-Greene coefficients! Why?

Wrong way map and Stanley functions

There is a $\mathbb{Q}$-algebra map η_0

$$\begin{array}{ccc} H^*(\mathrm{Fl}_{\mathbb{Z}}) & \xrightarrow{\eta_0} & H^*(\mathrm{Gr}) \\ \downarrow \cong & & \downarrow \cong \\ \Lambda(x_-) \otimes \mathbb{Q}[x] & \xrightarrow[\eta_0]{} & \Lambda(x_-) \end{array}$$

$$x_k \mapsto 0$$

$$p_r(x_-) \mapsto p_r(x_-)$$

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Definition

For $w \in S_{\mathbb{Z}}$ the Stanley symmetric function is

$$F_w = \eta_0(\overleftarrow{\mathfrak{S}}_w) \in \Lambda(x_-).$$

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$$F_{w_\lambda} = \overleftarrow{\mathfrak{S}}_{w_\lambda} = s_\lambda(x_-)$$

Edelman-Greene coefficients j^w_λ

Define $j^w_\lambda \in \mathbb{Q}$ by

$$F_w = \sum_{\lambda \in \mathbb{Y}} j^w_\lambda s_\lambda$$

Theorem [Edelman-Greene] $j^w_\lambda \in \mathbb{Z}_{\geq 0}$ with combinatorial formula.

Other formulas: [Lascoux,Schützenberger], [Haiman], [Reiner, S.]

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[Lam, SJ Lee, S.] New formula for j_λ^w using **bumpless pipedreams**

Hopf structure

$\Lambda(x_-)$ is a Hopf algebra with primitive generators p_r:

$$\Delta(p_r) = p_r \otimes 1 + 1 \otimes p_r \qquad \text{for all } r \geq 1.$$

$\overleftarrow{R} = \Lambda(x_-) \otimes \mathbb{Q}[x]$ is a $\Lambda(x_-)$-comodule: Act by Δ on $\Lambda(x_-)$ factor

$$\begin{array}{ccc}
\overleftarrow{R} & \xrightarrow{\Delta \otimes \mathrm{id}_{\mathbb{Q}[x]}} & \Lambda(x_-) \otimes \overleftarrow{R} \\
\Big\downarrow{=} & & \Big\downarrow{=} \\
\Lambda(x_-) \otimes \mathbb{Q}[x] & \xrightarrow[\Delta \otimes \mathrm{id}_{\mathbb{Q}[x]}]{} & \Lambda(x_-) \otimes \Lambda(x_-) \otimes \mathbb{Q}[x]
\end{array}$$

Coproduct formulae [Lam, SJ Lee, S.]

$w \doteq uv$ means $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$

Theorem

For all $w \in S_{\mathbb{Z}}$

$$(\Delta \otimes \mathrm{id}_{\mathbb{Q}[x]})(\overleftarrow{\mathfrak{S}}_w) = \sum_{\substack{w \doteq uv \\ (u,v) \in S_{\mathbb{Z}} \times S_{\mathbb{Z}}}} F_u(x_-) \otimes \overleftarrow{\mathfrak{S}}_v \qquad \textit{in } \Lambda(x_-) \otimes \overleftarrow{R}$$

$$\overleftarrow{\mathfrak{S}}_w = \sum_{\substack{w \doteq uv \\ (u,v) \in S_{\mathbb{Z}} \times S_{\neq 0}}} F_u(x_-) \mathfrak{S}_v \qquad \textit{in } \overleftarrow{R}$$

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Corollary

$$\overleftarrow{\mathfrak{S}}_w = \sum_{\substack{w \doteq uv \\ v \in S_{\neq 0} \\ \lambda \in \mathbb{Y}}} j^u_\lambda s_\lambda(x_-) \mathfrak{S}_v$$

$\overleftarrow{\mathfrak{S}}$-structure constants

For $u, v, w \in S_{\mathbb{Z}}$ (resp. S_+) define $\overleftarrow{c}\,^w_{uv} \in \mathbb{Q}$ (resp. c^w_{uv}) by

$$\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v = \sum_w \overleftarrow{c}\,^w_{uv} \overleftarrow{\mathfrak{S}}_w$$

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_w c^w_{uv} \mathfrak{S}_w.$$

Example

$$\overleftarrow{\mathfrak{S}}^2_{s_1} = \overleftarrow{\mathfrak{S}}_{s_2 s_1} + \overleftarrow{\mathfrak{S}}_{s_0 s_1} \qquad \mathfrak{S}^2_{s_1} = \mathfrak{S}_{s_2 s_1}$$

$$\overleftarrow{\mathfrak{S}}^2_{s_2} = \overleftarrow{\mathfrak{S}}_{s_3 s_2} + \overleftarrow{\mathfrak{S}}_{s_1 s_2} \qquad \mathfrak{S}^2_{s_2} = \mathfrak{S}_{s_3 s_2} + \mathfrak{S}_{s_1 s_2}$$

Proposition (Lam, SJ Lee, S.)

1. *For* $u, v, w \in S_+$, *we have* $c_{uv}^{w} = \overleftarrow{c}_{uv}^{w}$.
2. *Every back stable Schubert structure constant is a usual Schubert structure constant.*

Theorem (Lam, SJ Lee, S.)

Let $u \in S_m$, $v \in S_n$ *and* $\lambda \in \mathbb{Y}$. *Let* $u \times v := u\,\mathrm{shift}^m(v) \in S_{m+n} \subset S_+$. *Then*

$$j_\lambda^{u\times v} = \sum_{w \in S_{\mathbb{Z}}} \overleftarrow{c}\,^w_{uv} j_\lambda^w.$$

[1994 Reiner, S. unpublished] Explicit combinatorial conjecture for

$$\begin{aligned}\langle \eta_0(\mathfrak{S}_u\mathfrak{S}_v)\,,\, s_\lambda\rangle &= \langle \eta_0(\sum_{w\in S_+} c^w_{uv}\mathfrak{S}_w)\,,\, s_\lambda\rangle \\ &= \langle \sum_{w\in S_+} c^w_{uv} F_w\,,\, s_\lambda\rangle \\ &= \sum_{w\in S_+} c^w_{uv} j^w_\lambda\end{aligned}$$

Tells difference between c^w_{uv} and $\overleftarrow{c}\,^w_{uv}$.

Summary of novel features of infinite setting

- Back-stable Schubert basis
- Wrong way algebra map $H^*(\mathrm{Fl}) \to H^*(\mathrm{Gr})$ gives natural definition of Stanley function
- Negative Schubert polynomials
- Coproduct formula
- New clue on product of Schubs
- New formula for Edelman-Greene coefficients

Equivariance

$$H_T^*(\mathrm{pt}) \cong \mathbb{Q}[a] = \mathbb{Q}[a_k \mid k \in \mathbb{Z}]$$

$$H_T^*(\mathrm{Fl}) \cong H_T^*(\mathrm{Gr}) \otimes_{H_T^*(\mathrm{pt})} H_T^*(\mathrm{Fl}_- \times \mathrm{Fl}_+)$$

$$\begin{aligned} H_T^*(\mathrm{Fl}_- \times \mathrm{Fl}_+) &\cong H_T^*(\mathrm{pt}) \otimes H^*(\mathrm{Fl}_- \times \mathrm{Fl}_+) \\ &\cong \mathbb{Q}[x, a] \end{aligned}$$

Supersymmetric functions

Let $\Lambda(x||a)$ be the $\mathbb{Q}[a]$-algebra of supersymmetric functions:

$$\Lambda(x||a) = \mathbb{Q}[a][p_k(x||a) \mid k \geq 1]$$
$$p_k(x||a) = \sum_{i \leq 0} (x_i^k - a_i^k).$$

It is a polynomial Hopf $\mathbb{Q}[a]$-algebra generated by $p_k(x||a)$ for $k \geq 1$.
The $p_k(x||a)$ are primitive.

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Superization: the $\mathbb{Q}[a]$-algebra map

$$\mathbb{Q}[a] \otimes \Lambda(x_-) \to \Lambda(x||a)$$
$$p_k(x_-) \mapsto p_k(x||a)$$
$$f(x) \mapsto f(x/a)$$

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The $p_k(x||a)$ are primitive.

Superization: the $\mathbb{Q}[a]$-algebra map

$$\mathbb{Q}[a] \otimes \Lambda(x_-) \to \Lambda(x||a)$$
$$p_k(x_-) \mapsto p_k(x||a)$$
$$f(x) \mapsto f(x/a)$$

$$H_T^*(\mathrm{Gr}) \cong \Lambda(x||a)$$
$$H_T^*(\mathrm{Fl}) \cong \Lambda(x||a) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[x,a] =: \overleftarrow{R}(x;a)$$

Localization

$S_{\mathbb{Z}}$ acts on all x variables in $\overleftarrow{R}(x;a) = \Lambda(x||a) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[x,a]$:

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Definition

For $f \in \overleftarrow{R}(x;a) = \Lambda(x||a) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[x,a]$ and $w \in S_{\mathbb{Z}}$ define i_w^* by

$$\begin{array}{ccc} H^*(\mathrm{Fl}) & \xrightarrow{i_w^*} & H^*(\mathrm{pt}) \\ \cong \downarrow & & \downarrow \cong \\ \Lambda(x||a) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[x,a] & \xrightarrow[i_w^*]{} & \mathbb{Q}[a] \end{array}$$

$$i_w^*(f) = f(wa;a).$$

"Replace each x_i by $a_{w(i)}$" in both tensor factors.

Double Schubert polynomials [Lascoux,Schützenberger]

Definition

For $w \in S_n \subset S_+$ the double Schubert polynomial $\mathfrak{S}_w(x;a) \in \mathbb{Q}[x_+, a_+]$ is defined by

$$\mathfrak{S}_{w_0^{(n)}}(x;a) = \prod_{i+j \leq n} (x_i - a_j)$$

$$\mathfrak{S}_w(x;a) = \partial_i^x \mathfrak{S}_{ws_i}(x;a) \qquad \text{if } ws_i > w$$

It is well-defined for $w \in S_+$.

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$\mathfrak{S}_v(wa;a)$ is the localization of the v-th opposite Schubert class at the w-th T-fixed point.

"Negative" double Schubert polynomials

$\mathrm{rev}(s_i) = s_{-i} \qquad \mathrm{rev}(x_k) = -x_{1-k} \qquad \mathrm{rev}(a_k) = -a_{1-k}.$

For $w \in S_-$ define

$$\mathfrak{S}_w(x_-; a_-) = \mathrm{rev}(\mathfrak{S}_{\mathrm{rev}(w)}(x_+; a_+))$$

Example(s)

$$\mathfrak{S}_{s_3 s_2 s_1} = (x_1 - a_1)(x_1 - a_2)(x_1 - a_3)$$
$$\mathfrak{S}_{s_{-3} s_{-2} s_{-1}} = (-x_0 + a_0)(-x_0 + a_{-1})(-x_0 + a_{-2})$$

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For $w = w_- w_+ \in S_- \times S_+ = S_{\neq 0}$ define

$$\mathfrak{S}_w(x; a) = \mathfrak{S}_{w_-} \mathfrak{S}_{w_+}$$

$$\begin{aligned} H_T^*(\mathrm{Fl}_- \times \mathrm{Fl}_+) &\cong \mathbb{Q}[x; a] \\ &= \bigoplus_{w \in S_{\neq 0}} \mathbb{Q}[a] \mathfrak{S}_w(x; a) \end{aligned}$$

Back-stable double Schubert polynomials

$$\mathfrak{S}_w(x;a) = \sum_{w \doteq uv} (-1)^{\ell(u)} \mathfrak{S}_{u^{-1}}(a) \mathfrak{S}_v(x)$$

Since single Schubs back-stabilize, so do double Schubs.

Back-stable double Schubert polynomials

Define the back-stable double Schubert polynomial $\overleftarrow{\mathfrak{S}}_w(x;a)$ by

$$\overleftarrow{\mathfrak{S}}_w(x;a) = \sum_{\substack{w \doteq uv \\ u,v \in S_{\mathbb{Z}}}} (-1)^{\ell(u)} \overleftarrow{\mathfrak{S}}_{u^{-1}}(a) \overleftarrow{\mathfrak{S}}_v(x).$$

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$$\overleftarrow{\mathfrak{S}}_{\mathrm{shift}(w)}(x;a) = \mathrm{shift}(\overleftarrow{\mathfrak{S}}_w(x;a))$$

Proposition

$$\overleftarrow{\mathfrak{S}}_w(x;a) = \sum_{\substack{w \doteq uvz \\ u,z \in S_{\neq 0}}} (-1)^{\ell(u)} \mathfrak{S}_{u^{-1}}(a) F_v(x/a) \mathfrak{S}_z(x)$$

and in particular $\overleftarrow{\mathfrak{S}}_w(x;a) \in \overleftarrow{R}(x;a) = \Lambda(x||a) \otimes_{\mathbb{Q}[a]} \mathbb{Q}[x,a]$.

Double Stanley function (new!)

Wrong-way $\mathbb{Q}[a]$-algebra map η

$$\begin{array}{ccc} H_T^*(\mathrm{Fl}) & \longrightarrow & H_T^*(\mathrm{Gr}) \\ \downarrow & & \downarrow \\ \Lambda(x||a) \otimes \mathbb{Q}[x;a] & \xrightarrow[\eta]{} & \Lambda(x||a) \end{array}$$

$$f(x/a) \otimes g(x;a) \mapsto f(x/a)\, g(a;a)$$

Set x_i to a_i in the second tensor factor only.

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$$f(x/a) \otimes g(x;a) \mapsto f(x/a)\, g(a;a)$$

Set x_i to a_i in the second tensor factor only.

Definition

The **double Stanley function** $F_w(x||a) \in \Lambda(x||a)$ is defined by

$$F_w(x||a) = \eta(\overleftarrow{\mathfrak{S}}_w(x;a))$$

Theorem (Lam, SJ Lee, S.)

The equivariant class of Knutson's graph Schubert variety $[G(w)] \in H_T^*(\mathrm{Gr}(n, 2n))$ is given by $F_w(x||a)$ after truncation and setting $a_k \mapsto a_{k-n}$ for all k.

Equivariant coproduct formulae

Δ acts on $\Lambda(x||a)$ factor

Theorem (Lam, SJ Lee, S.)

$$\Delta(\overleftarrow{\mathfrak{S}}_w(x;a)) = \sum_{w \doteq uv} F_u(x||a) \otimes \overleftarrow{\mathfrak{S}}_v(x;a)$$

$$\overleftarrow{\mathfrak{S}}_w(x;a) = \sum_{\substack{w \doteq uv \\ v \in S_{\neq 0}}} F_u(x||a) \mathfrak{S}_v(x;a)$$

$$F_w(x||a) = \sum_{\substack{w \doteq uvz \\ u,z \in S_{\neq 0}}} (-1)^{\ell(u)} \mathfrak{S}_{u^{-1}}(a) F_v(x/a) \mathfrak{S}_z(a).$$

$$F_w(x||a) = \sum_{\substack{w \doteq uvz \\ u,z \in S_{\neq 0}}} (-1)^{\ell(u)} \mathfrak{S}_{u^{-1}}(a) F_v(x/a) \mathfrak{S}_z(a).$$

Take $w = w_\lambda$:
Durfee square $d(\lambda)$: Biggest $d \times d \subset \lambda$

Corollary

[Molev] [Lam, SJ Lee, S.]

$$s_\lambda(x||a) = \sum_{\substack{\mu \subset \lambda \\ d(\mu) = d(\lambda)}} (-1)^{|\lambda/\mu|} \mathfrak{S}_{w_{\lambda/\mu}^{-1}}(a) s_\mu(x/a)$$

$$s_\lambda(x/a) = \sum_{\substack{\mu \subset \lambda \\ d(\mu) = d(\lambda)}} \mathfrak{S}_{w_{\lambda/\mu}}(a) s_\mu(x||a)$$

Equivariant positivity, a la Peterson

Define the double (factorial) Schur functions $s_\lambda(x||a) \in \Lambda(x||a)$ by

$$s_\lambda(x||a) = \overleftarrow{\mathfrak{S}}_{w_\lambda}(x;a) \qquad \text{for } \lambda \in \mathbb{Y}.$$

Define the double Edelman-Greene coefficients $j_\lambda^w(a) \in \mathbb{Q}[a]$ by

$$F_w(x||a) = \sum_{\lambda \in \mathbb{Y}} j_\lambda^w(a) s_\lambda(x||a).$$

Examples

$$F_{s_{k+1}s_k}(x||a) = s_2(x||a) + (a_1 - a_{k+1})s_1(x||a)$$
$$F_{s_{k-1}s_k}(x||a) = s_{11}(x||a) + (a_k - a_0)s_1(x||a)$$

for all $k \in \mathbb{Z}$.

Theorem (Lam, SJ Lee, S.)

$$j_\lambda^w(a) \in \mathbb{Z}_{\geq 0}[a_i - a_j \mid i \prec j]$$

where

$$1 \prec 2 \prec 3 \prec \cdots \prec -3 \prec -2 \prec -1 \prec 0.$$

Problem

Find a combinatorial formula for $j_\lambda^w(a)$ *that exhibits this positivity.*

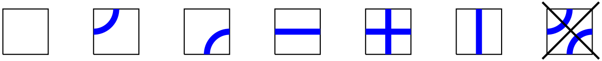
By Peterson's quantum = affine theorem, these are some of the equivariant Gromov-Witten invariants for Fl_n.

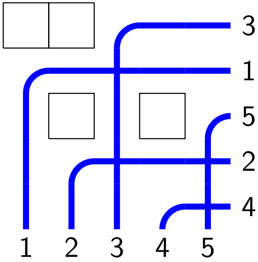
Summary for infinite equivariant setting

- Double Stanley functions and equivariant graph Schubert classes
- Equivariant positivity of Edelman-Greene coefficients
- Triple product formula for double Stanleys and back stable double Schuberts
-

Bumpless pipedreams

For S_n, tile $n \times n$ square by unit square tiles

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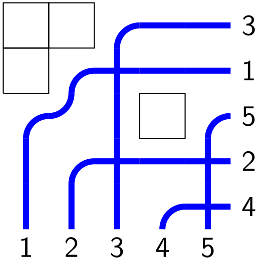
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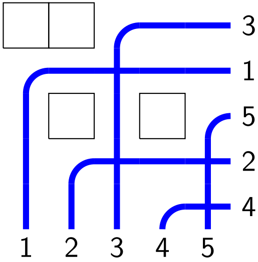
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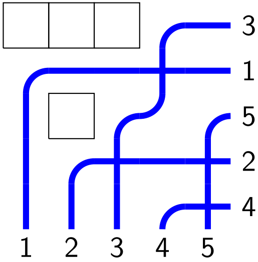
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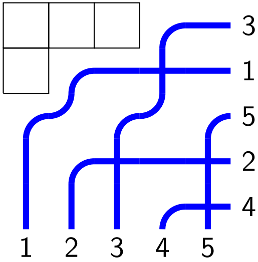
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Weight of box in row i and column j is $x_i - a_j$
Weight of pipedream is product of weights of boxes

Theorem (Lam, SJ Lee, S.)

- *The weighted sum of bumpless pipedreams for $w \in S_n$ inside the $n \times n$ box is $\mathfrak{S}_w(x; a)$.*
- *For $w \in S_{\mathbb{Z}}$ inside the plane: $\overleftarrow{\mathfrak{S}}_w(x; a)$.*
- *For $|\lambda| = \ell(w)$ the Edelman-Greene coefficient j_λ^w is the number of bumpless pipedreams for w of partition shape λ.*

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Different than [Billey, Bergeron] combinatorics

Homology $H_*^T(\mathrm{Gr})$, following [Molev]

- $H_T^*(\mathrm{Gr}) \cong \Lambda(x||a)$ and $H_*^T(\mathrm{Gr})$ are dual Hopf $\mathbb{Q}[a]$-algebras
- Let $\hat{\Lambda}(y||a)$ be the completion of symm. funcs. in $y = (\ldots, y_{-1}, y_0)$ with coefficients in $\mathbb{Q}[a]$. Degree is allowed to be unbounded
- $p_r(x||a)$ and $p_r(y)$ are primitive for $r \geq 1$
- $H_*^T(\mathrm{Gr})$ is isomorphic to a $\mathbb{Q}[a]$-subalgebra of $\hat{\Lambda}(y||a)$.
- There is a $\mathbb{Q}[a]$-bilinear pairing $\langle \cdot \, , \, \cdot \rangle : \Lambda(x||a) \otimes_{\mathbb{Q}[a]} \hat{\Lambda}(y||a) \to \mathbb{Q}[a]$:

$$\langle s_\lambda(x/a) \, , \, s_\mu(y) \rangle = \delta_{\lambda\mu}.$$

  That is, the pairing has reproducing kernel

$$\Omega[(x-a)y] = \prod_{i,j \leq 0} \frac{(1 - a_i y_j)}{(1 - x_i y_j)}$$

- Define Molev's dual Schur functions $\hat{s}_\lambda(y||a)$ by

$$\langle s_\lambda(x||a) \, , \, \hat{s}_\mu(y||a) \rangle = \delta_{\lambda\mu}.$$

$$s_\lambda(x||a) = \sum_{\mu\subset\lambda} (-1)^{|\lambda|-|\mu|} \mathfrak{S}_{w_\mu w_\lambda^{-1}}(a) s_\mu(x/a)$$

Corollary

$$\hat{s}_\mu(y||a) = \sum_{\substack{\lambda\supset\mu \\ d(\lambda)=d(\mu)}} \mathfrak{S}_{w_\lambda w_\mu^{-1}}(a) s_\lambda(y)$$

$$s_\mu(y) = \sum_{\substack{\lambda\supset\mu \\ d(\lambda)=d(\mu)}} (-1)^{|\lambda|-|\mu|} \mathfrak{S}_{w_\mu w_\lambda^{-1}}(a) \hat{s}_\lambda(y||a)$$

Example

$$\hat{s}_1(y||a) = \sum_{p,q\geq 0} (-a_0)^q a_1^p s_{(p+1,1^q)}(y)$$

Left divided differences

∂_i^a: divided difference on a variables.
$\lambda \pm i$: partition λ with box added to (removed from) i-th diagonal

$$\partial_i^a s_\lambda(x||a) = -s_{\lambda - i}(x||a)$$

where the answer is 0 if λ does not have a removable box on the i-th diagonal.

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where the answer is 0 if λ does not have a removable box on the i-th diagonal. Reason:

$$\partial_i^a \overleftarrow{\mathfrak{S}}_w(x;a) = \begin{cases} -\overleftarrow{\mathfrak{S}}_{s_i w}(x;a) & \text{if } s_i w < w \\ 0 & \text{otherwise.} \end{cases}$$

Homology divided differences

$$\Omega[(x-a)y] = \prod_{i,j\leq 0} \frac{(1-a_i y_j)}{(1-x_i y_j)}$$

Definition

[Naruse for type C_∞] [Lam, SJ Lee, S.]

$$\delta_i = \Omega[(x-a)y]\partial_i^a \Omega[(a-x)y]$$

If $i \neq 0$, since $a = a_- = (\ldots, a_{-1}, a_0)$, ∂_i^a commutes with $\Omega[(a-x)y]$ and we get

$$\delta_i = \partial_i^a \qquad \text{if } i \neq 0$$

$$\begin{aligned}\delta_0 &= \Omega[(x-a)y](a_0-a_1)^{-1}(\mathrm{id}-s_0^a)\Omega[(a-x)y] \\ &= \alpha_0^{-1}\Omega[-a_0 y](\mathrm{id}-s_0^a)\Omega[a_0 y] \\ &= \alpha_0^{-1}(\mathrm{id}-\Omega[(a_1-a_0)y]s_0^a).\end{aligned}$$

Analogy: $\Omega[(a_1-a_0)y]$ is a translation by "highest coroot" $a_1 - a_0$

Theorem (Lam, SJ Lee, S.)

$$\partial_i^a \hat{s}_\lambda(y||a) = \hat{s}_{\lambda+i}(y||a)$$

where the answer is 0 *if* λ *does not have an addable box on the* i*-th diagonal. In particular*

$$\hat{s}_\lambda(y||a) = \delta_{w_\lambda}(1).$$

Example

$$\begin{aligned}\hat{s}_1(y||a) &= \delta_0(1) = (a_0 - a_1)^{-1}(\mathrm{id} - \Omega[(a_1 - a_0)y]s_0)(1) \\ &= (a_0 - a_1)^{-1}\left(1 - \prod_{k\le 0} \frac{1 - a_0 y_k}{1 - a_1 y_k}\right) \\ &= \sum_{p,q\ge 0} (-a_0)^q a_1^p s_{(p+1,1^q)}(y)\end{aligned}$$

Conjecture (Lam, SJ Lee, S.)

$H_*^T(\mathrm{Gr})$ is isomorphic to the $\mathbb{Q}[a]$-subalgebra of $\hat{\Lambda}(y||a)$ generated by elements of the form

$$\frac{\Omega[(a_i - a_j)y] - 1}{a_i - a_j} \qquad \text{for } i \neq j.$$

For affine root systems [Bezrukavnikov, Finkelberg, Mirkovic]

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For affine root systems [Bezrukavnikov, Finkelberg, Mirkovic]

Similar operators work for affine type A, creating double k-Schur functions.

Future directions

- Schubert bases in the Rees rings ($T \times \mathbb{C}^*$)
- Affine flag varieties.
- K-theory; K-analogue of Rees construction
- Affine Hecke analogues

Sage code

- Localization for $H_T^*(\mathrm{Fl})$
- Symmetric function code for $H_T^*(\mathrm{Gr})$
- Double Affine Hecke algebra