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Kashiwara Crystals of Type A in Low Rank

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The Problem

- The irreducible modules for the symmetric groups over $\mathbb{C}$ are labelled by partitions.
- Over a field F of characteristic p, the irreducible modules of the tower of group algebras are labelled by p-regular partitions.
- Take a element ξ in a field k with $1 + \xi + \cdots + \xi^{e-1} = 0$. For the tower of cyclotomic Hecke algebras over K, the irreducible modules are labelled by e-regular multipartitions.

The problem: For $e > 2$, we have only a recursive algorithm for constructing e-regular multipartitions.

Affine Lie Algebras of Type A

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- $\mathcal{G}$ - an affine Lie algebra of type A,
- Dynkin diagram a circle,
- $\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell$ - fundamental weights, $\ell = e - 1$
- $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ - simple roots,
- $\delta = \sum \alpha_i$ - the null root.
- $\mathbb{Q}_+ = \{\alpha | \alpha = \sum c_i \alpha_i\}$, with content $(c_0, c_1, \ldots, c_\ell)$,
- A corank 1 Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 & -1 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & -1 & 2 & -1 \\ -1 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix}$$

Kashiwara crystals

- Let $e_i, f_i, h_i, i = 0, 1, \ldots, \ell,\ c$ be a Chevalley basis
- Let $\Lambda = a_0\Lambda_0 + \ldots a_\ell\Lambda_\ell, a_i \in \mathbb{Z}_+$
- Let $V(\Lambda)$ be a highest weight representation generated by the f_i from $u_\emptyset$ of weight Λ
- Let $P(\Lambda)$ be the sets of weights of weight spaces of $V(\Lambda)$
- Let $\max(\Lambda)$ be the set of weights $\eta \in P(\Lambda)$ such that $\eta + \delta \notin P(\Lambda)$.

A Kashiwara *crystal* $B(\Lambda)$ is

- a labeling of the basis of $V(\Lambda)$
- operations $\tilde{e}_i$ and $\tilde{f}_i$,
- functions ϕ_i, ϵ_i measuring the distance to the end of the local i-string.

Kashiwara crystals of Type A

There are three important versions of the Kashiwara crystal in type A:

- by e-regular multipartitions
- by Littelmann paths,
- and by canonical basis elements, in a space called Fock space, with coefficients which are v-polynomials

Kashiwara crystals of Type A

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Our general research program concerns the combinatorial relations among all three, but for this talk, we focus on the possibility of passing directly between the e-regular multipartitions and the canonical basis elements. In the process, we also found a result on the Morita eqivalence classes of cyclotomic Hecke algebras.

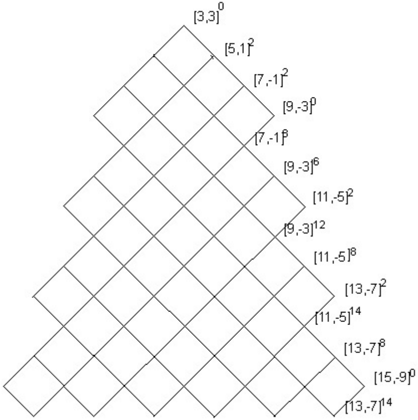
The reduced crystal

We get the *reduced* crystal [AS] with vertices $P(\Lambda)$ by adding edges wherever there is an edge in the underlying Kashwara crystal, where we take all i-strings parallel to each other. The weights in $P(\Lambda)$ are of the form $\lambda = \Lambda - \alpha$ for some $\alpha \in Q_+$. The highest-weight representation being integrable, all i-strings are of finite length. To each vertex of $P(\Lambda)$ we associate

- The content $(c_0, \ldots, c_\ell)$ of $\alpha = c_0\alpha_0 + \cdots + c_\ell\alpha_\ell$
- The defect $\mathrm{def}(\lambda) = (\Lambda \mid \alpha) - \frac{1}{2}(\alpha \mid \alpha)$
- The hub $\theta = (\theta_0, \ldots, \theta_\ell)$, where $\theta_i =< h_i, \lambda >$

The vertices of defect zero are those equivalent to Λ under the action of the Weyl group W.

Reduced crystal, $e = 2, \Lambda = 3\Lambda_0 + 3\Lambda_1$, with hubs



The reduced crystal for $e = 2, \Lambda = 3\Lambda_0 + 3\Lambda_1$, truncated at degree 13

Categorification in Type A

Chuang and Rouquier [CR] categorification:

- $V(\Lambda) \Leftarrow \oplus Mod(H_n^\Lambda)$, $n = 0, 1, 2, \ldots$.
- $e_i, f_i \Leftarrow$ restriction and induction functors E_i, F_i
- Weight spaces $\Leftarrow$ blocks,
- $s_i \in W \Leftarrow$ derived equivalences,
- s_i acting on end-points of i-string $\Leftarrow$ Morita equivalence
- $b \in B(\Lambda) \Leftarrow$ simple modules of H_n^Λ

The simplest but best known example is for $r = 1$, $\Lambda = \Lambda_0$, over a field of characteristic e, where the simple modules of the symmetric groups correspond to e-regular partitions.

The Fundamental Region

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We want to show that for a given defect d there are only a finite number of Morita equivalence classes of cyclotomic Hecke algebras, relying on categorification and using combinatorics. So far we can to this for ranks $e = 2, 3$. We start with the following theorem.

Theorem (Barshavsky, Fayers, S., 2013) For any $\eta = \Lambda - \sum_{i=0}^{\ell} b_i \alpha_i$ there is a unique s such that

$$\eta - s\delta \in max(\Lambda)$$

The proof, which is given for any type of affine Lie algebra, depends on finding a fundamental region in $P(\Lambda)$ from which every element of $\max(\Lambda)$ can be reached by transformations in the normal abelian subgroup T in the decomposition of the Weyl group as

$$W = T \rtimes W^{\circ}$$

The Fundamental Region

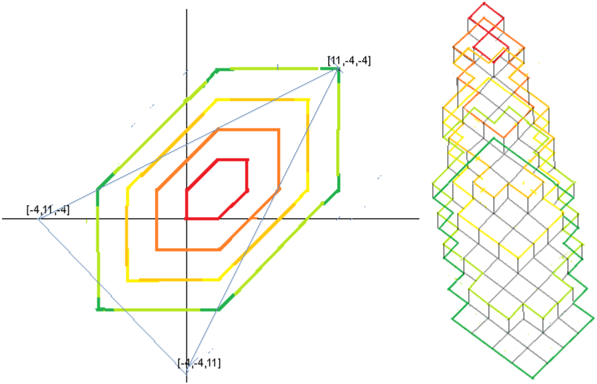
The elements of T are transformations of the form

$$t_\alpha(\zeta) = \zeta + r\alpha - ((\zeta|\alpha) + \frac{1}{2}(\alpha|\alpha)r)\delta$$

By the theorem in the previous slide, every vertex in max(Λ) is equivalent by the action of the Weyl subgroup T to a point in the fundamental region and every defect is congruent mod r to a defect in the fundamental region. Thus the elements of $max(\Lambda)$ correspond one-to-one to points of the integral lattice generated by $\alpha_1, \ldots, \alpha_\ell$.

On a string, def$(\lambda - k\alpha_i)$ is parabolic in k, so the defects rise to the center, and for a string of length c there can be no defect less than c except at the ends. Lengths of i-strings are determined by θ_i

Lattice and crystal



The reduced crystal for $e = 3, \Lambda = \Lambda_0 + \Lambda_1 + \Lambda_2$

Maximal Strings

Proposition

For any defect d in a rank 2 or 3 crystal, there is a degree $N(d)$ such that every occurence of that defect in degree more than $N(d)$ is at the end of a string to a vertex of lower degree.

Proof.

Case $e = 3$: Let $c = r + 2d$. Let N be the maximal degree occuring in a triangle $[c, -d, -d]$, $[-d, c, -d]$, $[-d, -d, c]$. Every hub on or outside the triangle contains a negative θ_i with $\theta_i \leq -d$. Every vertex inside the triangle has degree lower than $N + 1$. Let $\overline{d}$ be the residue of d mod r and set $N(d) = N + 1 + (d - \overline{d})$. Thus every $b \in B(\Lambda)$ of defect d lies on a string leading to a lower degree of length $\geq d$. Since a basis element of defect d cannot be internal, it is at the end of a this string.

Multipartitions

The canonical basis elements correspond to e-regular multipartitions. In order to generate the e-regular multipartitions, we must choose an ordering of the fundamental weights in Λ,

$$\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_r}$$

We will follow Mathas in [M] in requiring $k_1 \leq k_2 \leq \cdots \leq k_r$, where the number of terms, r, in the sum is the level. We can then summarize by setting

$$\Lambda = a_0\Lambda_0 + \cdots + a_\ell\Lambda_\ell$$

The Young diagram of a defect 0 weight λ will be represented by $Y(\lambda)$. If the m-th partition of λ is nonempty, then we associate to each node in the Young diagram a residue, where the node (i, j) is given residue

$$k_m + j - i$$

Involutions of multipartitions

In [Fa], Fayers describes two involutions on the multipartitions:

Definition

If $\lambda = (\lambda^1, \ldots, \lambda^r)$ is a multipartition of rank e and level r, then the *conjugate* λ' of λ is given by $\lambda' = (\lambda^{r\prime}, \ldots, \lambda^{1\prime})$, where $\lambda^{i\prime}$ is the transposed partition of λ^i , corresponding to reflection of the Young diagram in the main diagonal.

Definition

If $\lambda = (\lambda^1, \ldots, \lambda^r)$ is a multipartion of rank e and level r for $\Lambda = \Lambda_{k_1} + \ldots \Lambda_{k_r}$, then the *diamond* $\lambda^\diamond$ of λ is a multipartition in the crystal for $\hat{\Lambda} = \Lambda_{-k_r} + \ldots \Lambda_{-k_1}$, whose path is obtained from a path giving λ by replacing each residue by minus that residue.

Fock Space

- $\mathcal{U} = \mathcal{U}_v(\hat{s}l(e))$: quantum enveloping algebra over $\mathbb{Q}(v)$
- $[n]_v = v^{n-1} + v^{n-3} + \cdots + v^{-(n-3)} + v^{-(n-1)}$. .
- Generators: e_i, f_i, h_i for $i \in I = \mathbb{Z}/\mathbb{Z}e$ and a central c.
- If $\Lambda = \Lambda_{k_1} + \cdots + \Lambda_{k_r}$, set $s = (k_1, \ldots, k_r)$.

The Fock space $\mathcal{F}^s$ is a space with basis given by multipartitions consisting of r partitions. An addable i-node $\mathfrak{n}$ is a node outside λ such that if added it would give a multipartition $\lambda^{\mathfrak{n}}$ and would have residue i. A removable i-node $\mathfrak{m}$ inside a multipartition μ is a node at the end of a row or column which would give a multipartition $\mu_{\mathfrak{m}}$ if removed.

Fock Space

The quantum enveloping algebra $\mathcal{U}_v(\hat{s}l(e))$ acts on the Fock space by determining actions for the elements of the Chevalley basis, as follows:

- For an addable node, define $N(\mathfrak{n}, i) = \#\{$ addable i-nodes above $\mathfrak{n}\} - \#\{$ removable i-nodes above $\mathfrak{m}\}$ and set

$$f_i(\lambda) = \sum_{\mathfrak{n}} v^{N(\mathfrak{n},i)} \lambda^{\mathfrak{n}}.$$

- For a removable node, define $M(\mathfrak{m}, i) = \#\{$ addable i-nodes below $\mathfrak{m}\} - \#\{$ removable i-nodes below $\mathfrak{m}\}$.

$$e_i(\mu) = \sum_{\mathfrak{m}} v^{M(\mathfrak{m},i)} \mu_{\mathfrak{m}}.$$

Fock Space

The divided powers are $e_i^{(k)}$ and $f_i^{(k)}$ and they are given by dividing by the quantum factorials $[k]_v!$. We define $\mathcal{F}^s_{\mathcal{A}}$ to be the subalgebra of $\mathcal{F}^s$ generated by the divided powers from the highest weight vector over $\mathcal{A}$, where coefficients lie in the algebra $\mathcal{A}$ of Laurent polynomials in v with integral coefficients. In addition, there is an involution of the quantum enveloping algebra called the bar-involution which fixes e_i, f_i and h_i, but interchanges v and v^{-1}.

Canonical basis

For each e-regular multipartition μ, there is an element $G(\mu)$ of the Fock space $\mathcal{F}^s_{\mathcal{A}}$ that is invariant under the bar involution. and these are called the canonical basis elements. The action of the Chevalley basis elements e_i and f_i on these canonical basis elements is induced from their action on the basis elements of the Fock space. We write

$$G(\mu) = \sum_{\lambda \in P^r} d_{\lambda\mu}(v)\lambda$$

Definition

For a sequence S chosen from a two element ordered set $\{0, 1\}$, the number of inversions, $\mathrm{Inv}(S)$, is the sum of the number of elements 0 appearing before each element 1.

Definition

The *shape* of a canonical basis element is the number of multipartitions, counting repetitions, for each power of v.

Definition

A vertex v in the reduced crystal is i-external if $\tilde{e}_i(b)$ is zero for every element of $B(\Lambda)$ with the weight corresponding to v.

External canonical basis elements

Lemma

Let μ be an e-regular multipartition for an i-external vertex of the reduced crystal with i-string of length c. Let $\mathcal{S}(c,k)$ be the set of sequences of length c with k copies of 1 and $c-k$ copies of 0. For any $S \in \mathcal{S}(c,k)$, let λ^S be the multipartition obtained be adding each node corresponding to a 1. If every multipartitions occuring in $G(\lambda)$ has c of addable i-nodes and no removable nodes, then

$$\tilde{f}_i^{(k)}(G(\mu)) = \sum_{S \in \mathcal{S}(c,k)} \sum_{\lambda \in P^r} d_{\lambda\mu} v^{Inv(S)} \lambda^S$$

For $k = c$ and S the unique sequence with all copies of 1,

$$G(\mu^S) = \sum_{\lambda \in P^r} d_{\lambda\mu} \lambda^S$$

Symmetric Crystals

In Theorem 2.1 of [Fa], Fayers proves that if $w(\mu)$ is the defect of an e-regular multipartition μ, then

$$\hat{d}_{\lambda'\mu^\diamond} = v^{w(\mu)} d_{\lambda\mu}(v^{-1}).$$

This theorem involves constructing two distinct crystals and comparing them. Instead, let $\Lambda = a\Lambda_0 + a\Lambda_1 + \cdots + a\Lambda_\ell$ be symmetric. By using Fayer's $\diamond$ involution, followed by a reshuffling of the tuple s used in the definition of the Fock space, we find that for any path, the path we get by exchanging k with $\ell - k$ has canonical basis elements which by symmetry have the same shape and by the $\diamond$ involution are thus symmetric.

Results for Symmetric Crystals

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Let us take a completely symmetric crystal,
$\Lambda = a\Lambda_0 + a\Lambda_1 + \cdots + a\Lambda_\ell$.

- The canonical basis elements of a symmetric crystal have a symmetric shape.
- For $e = 2$, we can give formulae for the canonical basis elements of all weights with defect $a(k - a)$
- For $e = 3$, we can give formula for canonical basis elements with for weights $\lambda - k\alpha_i$ with $0 \leq k \leq a$. Every block of the cyclotomic Hecke algebra is Morita equivalent to one of a finite set of blocks of degree less than some bound $N(d)$ and the canonical basis elements are easily obtained from those of this finite set.

Credits: Computer resources for crystals

The Littelmann paths for a given $\mathcal{G}$ and Λ can be generated in Sagemath using the function CrystalOfLSPaths() written by Mark Shimozono and Anne Schilling. In addition, Travis Scrimshaw implemented an algorithm of Matt Fayers to calculate the canonical basis, named FockSpace().
Our own modification computes the following for a basis element $b \in B(\Lambda)$:

- The multipartition
- The Littelmann path and, optionally, the corner-points
- The canonical baisis element
- The set of paths in the reduced crystal leading to b

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Thank you.

Thank you.

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