Chained Permutations

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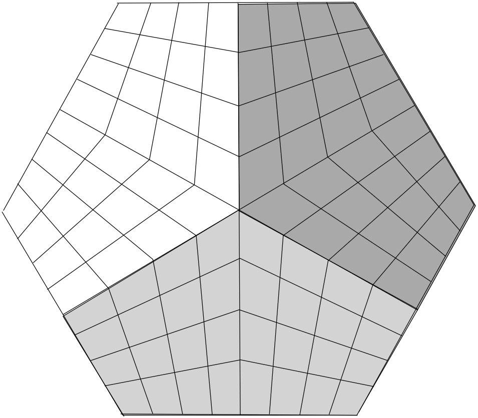
Three person chessboard



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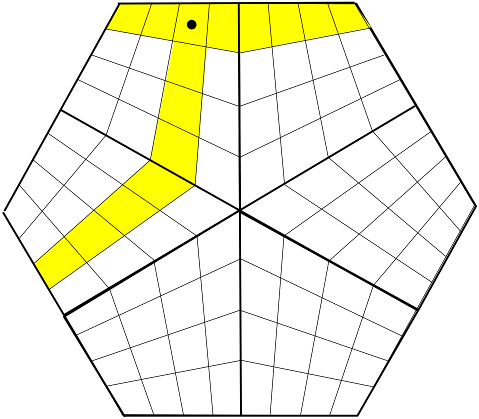

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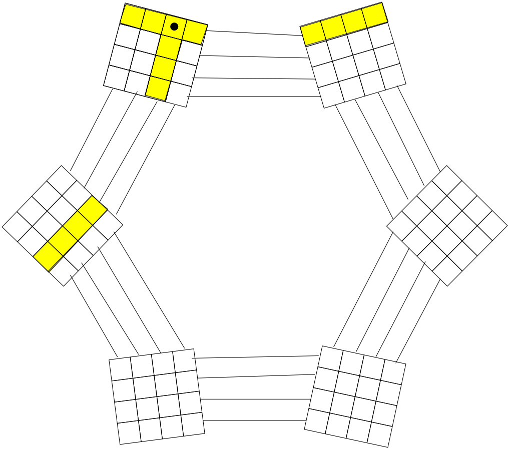
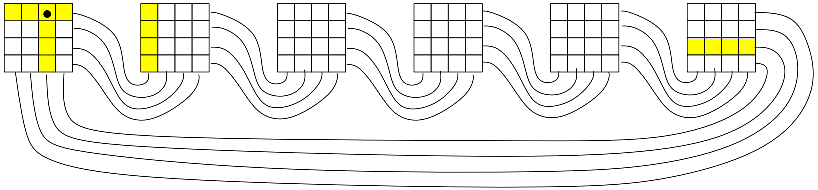
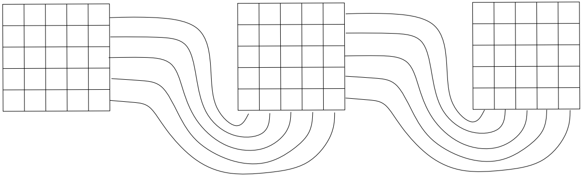


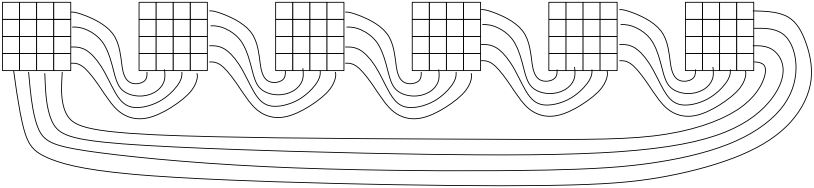
Three person chessboard - Rearranged



Two new families of chessboards



The board $B^{-}_{5,3}$



The board $B^{\circ}_{4,6}$

General enumerative result

Theorem

The number of ways to place m non-attacking rooks on board $B \in \{B^-_{n,k}, B^\circ_{n,k}\}$ is

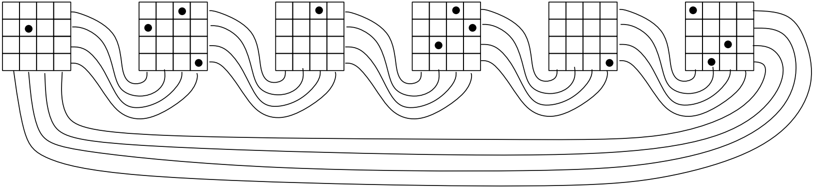
$$\sum_{(a_1,\ldots,a_k)\in\mathfrak{C}_m(B)} \prod_{i=1}^{k} \binom{n-a_{i-1}}{a_i} (n)_{a_i}$$

where a_0 is defined as:

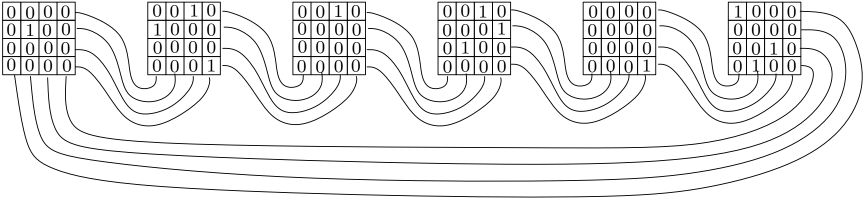
$$a_0 = \begin{cases} 0 & \textit{if } B = B^-_{n,k} \\ a_k & \textit{if } B = B^\circ_{n,k}. \end{cases}$$

Chained permutations

Maximum rook placement:

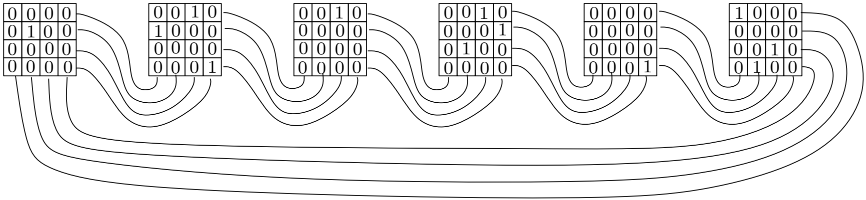


Permutation matrix form:



Chained permutations

Permutation matrix form:



One-line notation:

$$0200 - 3104 - 3000 - 3420 - 0004 - 1032 -$$

Work towards an analog of Bruhat order

- With usual permutations, we can use adjacent transpositions to obtain weak order.

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- Thinking of a permutation in matrix form, we can think of an adjacent transposition as swapping adjacent rows.
- There is a "natural" way to modify this in the case of chained permutations.
- We can perform a swap of adjacent rows on the ith matrix, while simultaneously performing a corresponding swap of adjacent columns on the $(i+1)$st matrix.

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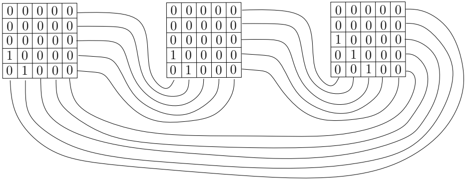
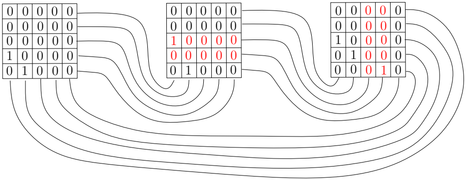


00012 – 00012 – 00123–

$$\downarrow s_{3,2}$$



00012 – 00102 – 00124–

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- SageMath has been useful not only for its computational power, but also for its ability to visualize and work with graphs and posets.

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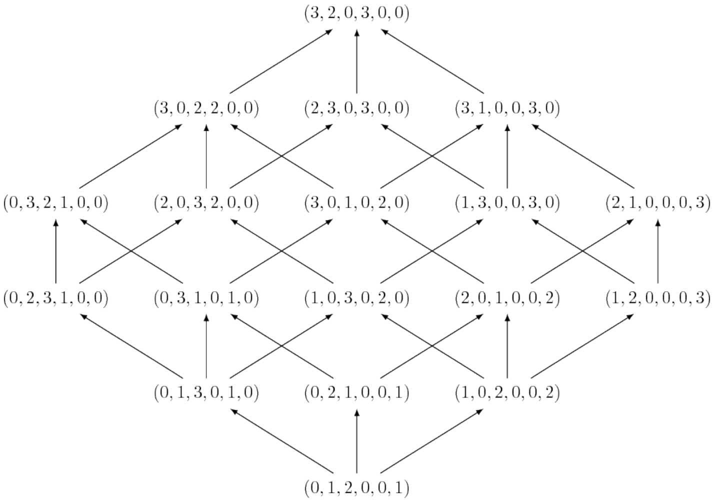


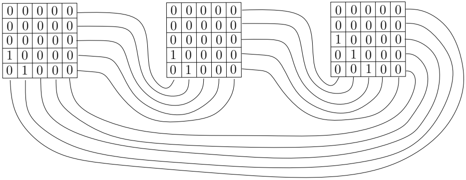
$k = 2$, $n = 3$, circular, fixed composition (2,1)

Work towards an analog of Bruhat order

Inversion number?

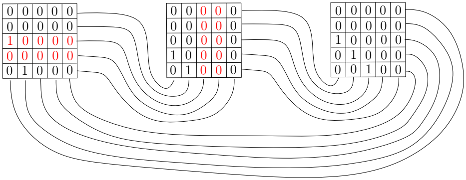
- It seems that there is a relatively "nice" analog of inversion number for chained permutations.

Work towards an analog of Bruhat order



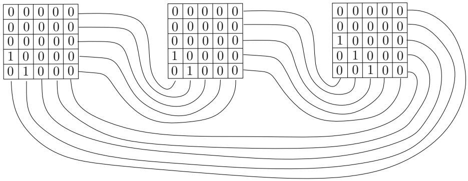
00012 – 00012 – 00123–

$\downarrow s_{3,1}$



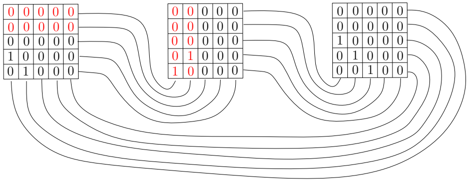
00102 – 00012 – 00123–

Work towards an analog of Bruhat order



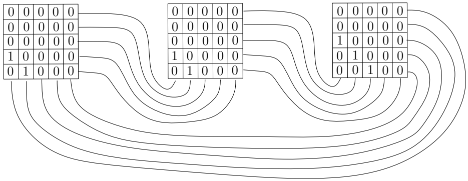
00012 – 00012 – 00123–

$\downarrow s_{1,1}$



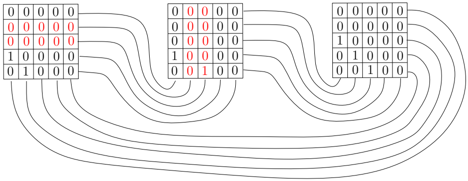
00012 – 00021 – 00123–

Work towards an analog of Bruhat order



00012 – 00012 – 00123–

$\downarrow s_{2,1}$



00012 – 00013 – 00123–

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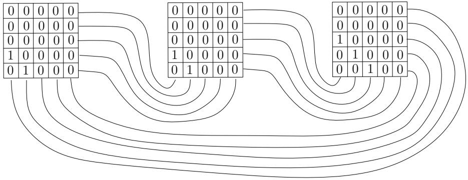
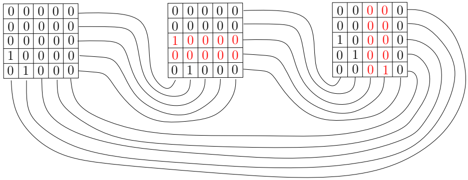


00012 – 00012 – 00123–

$\downarrow s_{3,2}$



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- It seems that there is a relatively "nice" analog of inversion number for chained permutations.
- We can start with a chained permutation and algorithmically change it to the identity.
- In fact, it appears to be the case that using this analog of inversion number,

$$\sum_{w \in P_{n,k}} q^{inv(w)} = \prod_{i=1}^{k} \begin{bmatrix} n - a_{i-1} \\ a_i \end{bmatrix}_q [n]_{a_i}$$

(the q-analog of the counting formula),

just as it is with usual permutations.

Thank you!