Differential Geometry $\rightsquigarrow$ Algebra $\rightsquigarrow$ Combinatorics (& back?)

SageDays @ ICERM – Providence 2018

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Supported by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01.0004).

EUROPSKI STRUKTURNI I INVESTICIJSKI FONDOVI

Operativni program KONKURENTNOST I KOHEZIJA

Europska unija Zajedno do fondova EU

Outline

1. Differential Geometry $\rightsquigarrow$ Algebra
2. Algebra $\rightsquigarrow$ Combinatorics
3. Combinatorics $+$ Algebra $\rightsquigarrow$ Differential Geometry?

Differential Geometry $\rightsquigarrow$ Algebra

Invariant differential operators

homogeneous space G/P, P-representation $\mathbb{V}$

$$\mathcal{V} = G \times_P \mathbb{V} \to G/P$$

$$\mathcal{D}\colon \Gamma^\infty(G/P, \mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$
$$\mathcal{D} \circ \widetilde{\rho_{\mathbb{V}}} = \widetilde{\rho_{\mathbb{W}}} \circ \mathcal{D}$$

$$D\colon \Gamma^\infty(G/P, \mathcal{J}^k\mathcal{V}) \to \Gamma^\infty(G/P, \mathcal{W})$$

everything is equivariant $\rightsquigarrow$ D is determined by germ at eP $\rightsquigarrow$ $\varphi : J^k\mathbb{V} \to \mathbb{W}$

Passing to dual maps and taking the limit $k \to \infty$ we get

$$\mathrm{Hom}_{\mathfrak{p}}\left(\mathbb{W}^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^*\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{W}^*, \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{V}^*\right)$$

Verma modules

G complex, P parabolic, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_+$, $\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{l} \oplus \mathfrak{p}_+$

$\lambda \in \mathfrak{h}^*$ which is $\mathfrak{l}$-dominant integral and hence defines finite-dimensional L-module $\mathbb{F}(\lambda)$

$$M(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{F}(\lambda)$$

Open problem:

$$\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) = ?$$

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Open problem:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) &= ?\\ &= \langle v \in M(\lambda) \mid \forall X \in \mathfrak{p}_+ \cup \mathfrak{n}_{\mathfrak{l}} : X \cdot v = 0 \rangle\end{aligned}$$

One way to find elements in $\mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda))$

$$M(\lambda) \simeq_{\mathfrak{g}} \mathrm{Pol}[\mathfrak{p}_+] \otimes \mathbb{F}(\lambda)$$

where the action of $\mathfrak{g}$ on polynomials is given by differential operators with polynomial coefficients.

homomorphisms of Verma modules are given by singular vectors $\rightsquigarrow$ system of PDEs on polynomials!

Algebra $\rightsquigarrow$ Combinatorics

BGG resolutions and Lie algebra (co)homology

$\lambda \in \mathfrak{h}^*$ $\mathfrak{g}$-integral, dominant $\rightsquigarrow L(\lambda)$ finite-dimensional $\mathfrak{g}$-module

affine action of W:

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

BGG resolution

$$\cdots \to \bigoplus_{\substack{w \in W^{\mathfrak{l}} \\ l(w)=i}} M(w \cdot \lambda) \to \cdots \bigoplus_{\substack{w \in W^{\mathfrak{l}} \\ l(w)=1}} M(w \cdot \lambda) \to M(\lambda) \to L_\lambda$$

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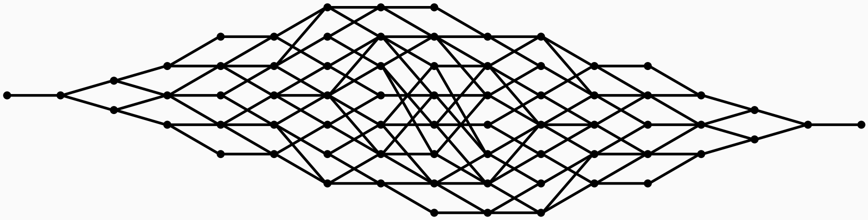
Kostant's theorem on niplotent cohomology

$$H^i(\mathfrak{p}_+, L_\lambda) = \bigoplus_{\substack{w \in W^{\mathfrak{l}} \\ l(w)=i}} \mathbb{F}_{w \cdot \lambda} = H_i(\mathfrak{p}_+, L_\lambda)$$

Nilpotent cohomology / BGG resolution for $\mathrm{SU}(2,2)$

$$
\begin{array}{ccccccccc}
 & & & & (2,\ -3,\ 0) & & & & \\
 & & & \nearrow & & \searrow & & & \\
(0,\ 0,\ 0) & \longrightarrow & (1,\ -2,\ 1) & & & & (1,\ -4,\ 1) & \longrightarrow & (0,\ -4,\ 0) \\
 & & & \searrow & & \nearrow & & & \\
 & & & & (0,\ -3,\ 2) & & & &
\end{array}
$$

The BGG graph of type $(A_7, A_3 \times A_3)$



Sage

Sage

For big parabolics much more efficient to use that $W^{\mathfrak{l}}$ parametrizes W-orbit of $\rho_{\mathfrak{l}}$.

Enright's formula

$\lambda \rightsquigarrow S_\lambda \subseteq \Phi(\mathfrak{p}_+)$

$\rightsquigarrow W_\lambda$ – subgroup of W which is generated by reflections s_α for $\alpha \in S_\lambda$

$\rightsquigarrow (\mathfrak{g}_\lambda, \mathfrak{p}_\lambda), \quad \mathfrak{p}_\lambda = \mathfrak{l}_\lambda \oplus \mathfrak{p}_{\lambda+}$

Theorem (3.7 of [DES91])
For unitarizable highest weight modules $L(\lambda)$ and for $i \in \mathbb{N}$ we have

$$H^i(\mathfrak{p}_+, L(\lambda)) \simeq \bigoplus_{\substack{w \in W_\lambda^{\mathfrak{c}} \\ l_\lambda(w)=i}} \mathbb{F}(\overline{w(\lambda+\rho)} - \rho)$$

where $\overline{\lambda}$ is the unique $\Phi_{\mathfrak{l}}^+$-dominant element in the $W_{\mathfrak{l}}$ orbit of λ and $W_\lambda^{\mathfrak{c}} = \{w \in W_\lambda \mid w\rho \text{ is } \Phi_{\mathfrak{l}_\lambda}^+\text{-dominant}\}$.

Deodhar, Dyer

For a Coxeter system (W, R) denote T to be the W-conjugates of R and let

$$N(w) = \{t \in T : l(tw) < l(w)\}.$$

If W' is a reflection subgroup of W, then

$$R' = \{t \in T : N(t) \cap W' = \{t\}\}$$

is a set of Coxeter generators for W' and (W', R') is a Coxeter system.

Combinatorics + Algebra $\rightsquigarrow$ Differential Geometry?

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INPUT: $(\mathfrak{g}, \mathfrak{p})$
OUTPUT: formulas for invariant differential operators

$$\begin{aligned} \lambda &\rightsquigarrow \mathbb{F}_\lambda \\ &\rightsquigarrow \mathfrak{g} \hookrightarrow \mathcal{A}_n \otimes \mathbb{F}_\lambda \\ &\rightsquigarrow \text{SageManifolds} \end{aligned}$$

Thank you for attention!

References

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