Shifted Symmetric Functions from Heisenberg Categories

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University of Southern California

July 26, 2018

joint work with Michael REEKS, Henry KVINGE

Outline

Categorification

Mathematicians do not study objects, but relations between objects.

Henri Poincaré

Heisenberg Algebra

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- *Induction* and *restriction* on symmetric group S_k representations:

$$\mathrm{Res}^{k}_{k+1}\,\mathrm{Ind}^{k+1}_{k} - \mathrm{Ind}^{k}_{k-1}\,\mathrm{Res}^{k-1}_{k} = \mathrm{Id}$$

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$$\bigoplus_k \underline{S_k - mod}$$

$\downarrow K_0$

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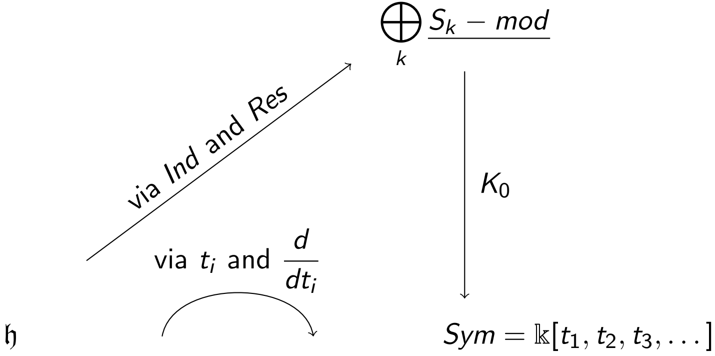

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Heisenberg Category?

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via *Ind* and *Res*

K_0

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$\mathfrak{h}$

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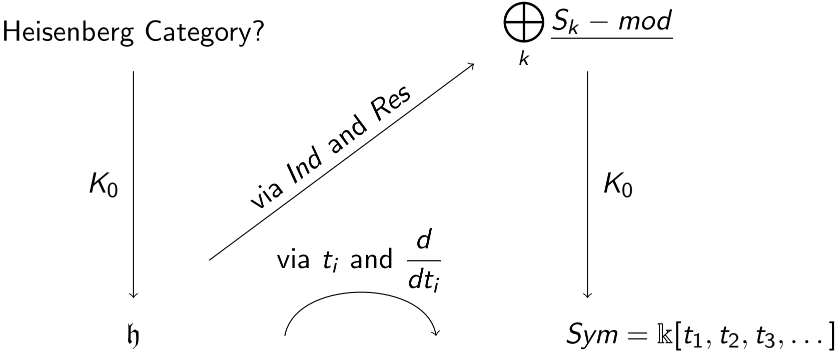

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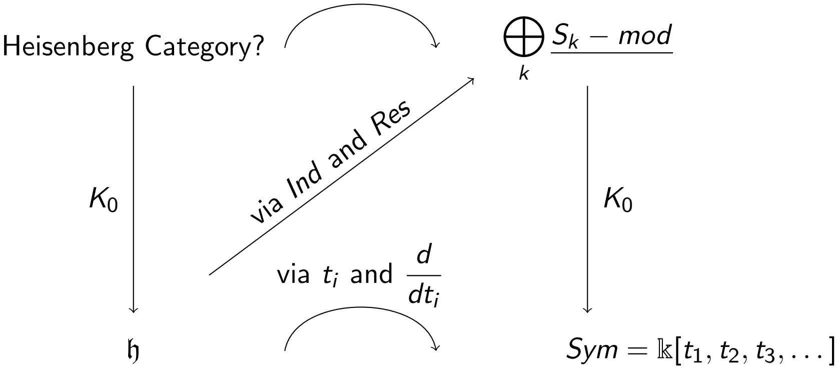

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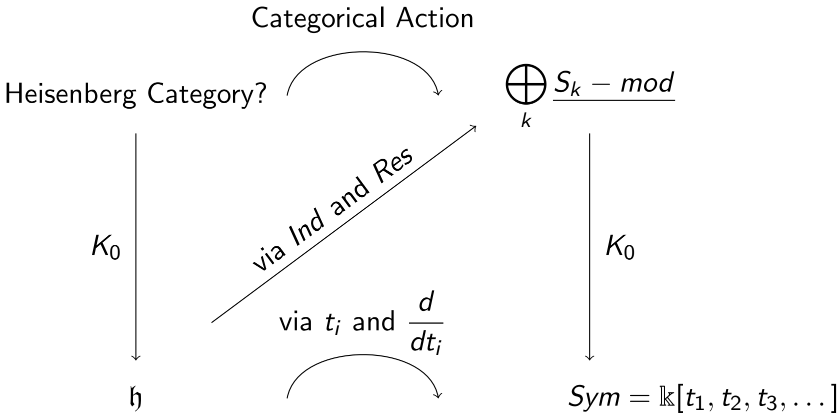

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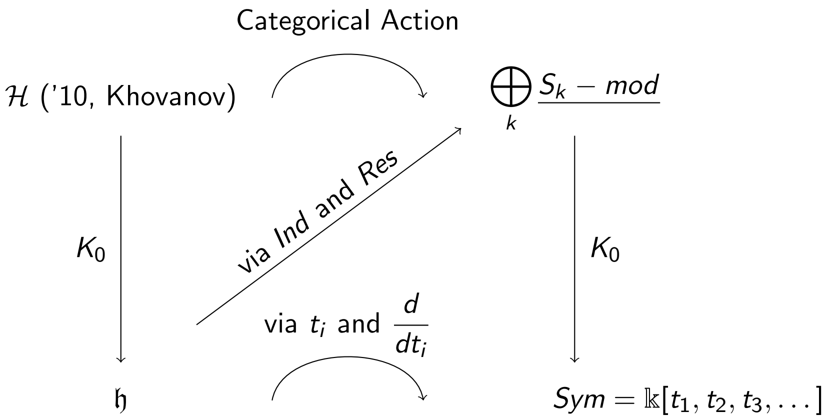

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$$(\mathfrak{h}, +, \times, 1) \longrightarrow (\mathcal{H}, \oplus, \otimes, \mathbb{1})$$

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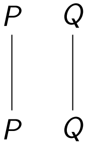

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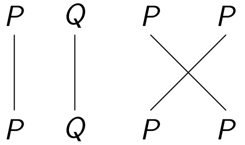

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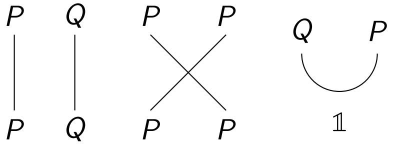

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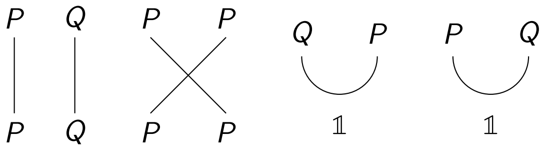

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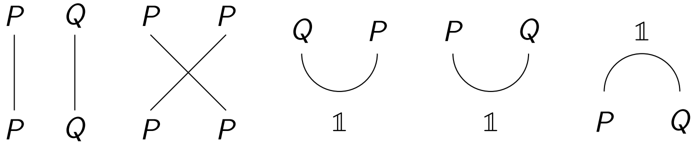

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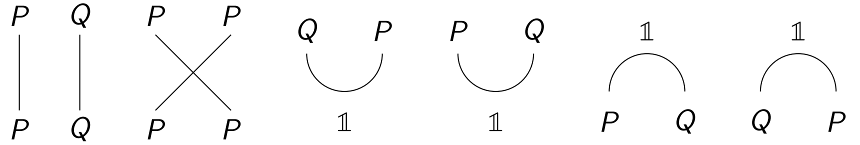

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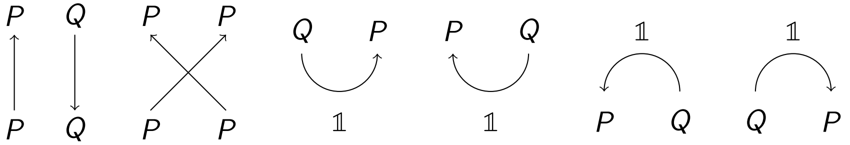

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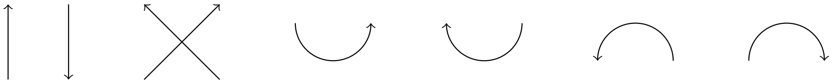

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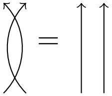

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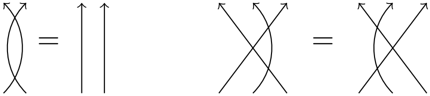

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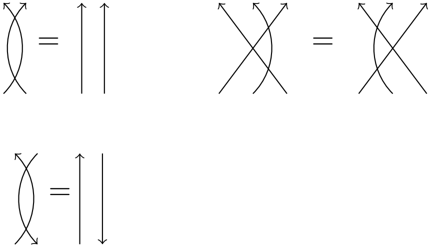

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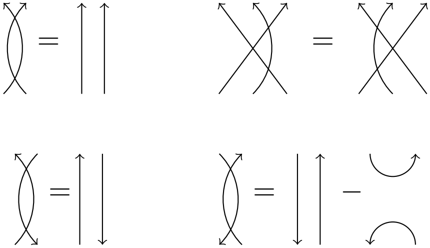

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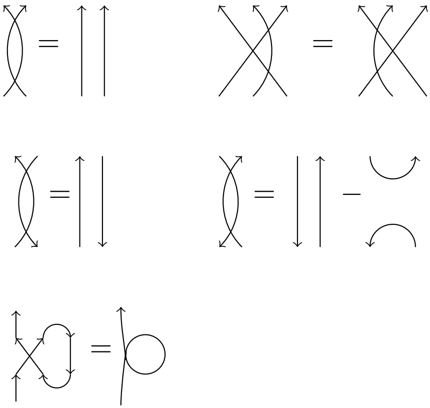

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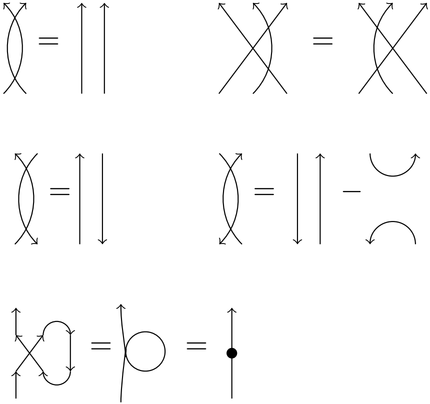

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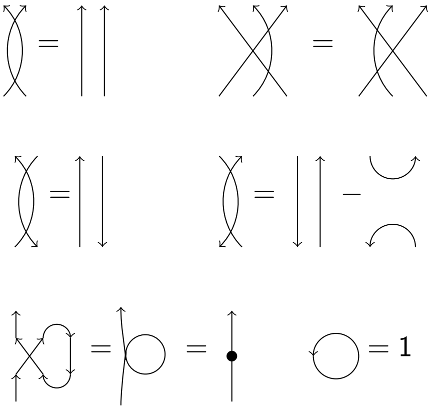

Outline

Categorical action of $\mathcal{H}$

$$\mathcal{H} \xrightarrow{\quad F_n \quad} \bigoplus_{k \geq n} (S_k, S_n)\text{-mod}$$

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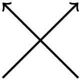 $\longrightarrow$ Bimodule map $S_{n+2} \mapsto S_{n+2}$

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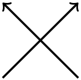
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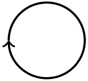

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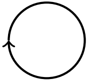

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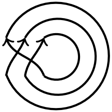

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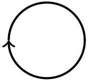 $\longrightarrow$ Bimodule map $S_n \mapsto S_n$

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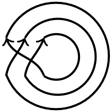 $\longrightarrow$ Central element in $\mathbb{C}[S_n]$

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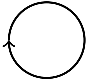

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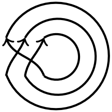

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Question: These closed diagrams correspond to which central elements?

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Indexed by partitions

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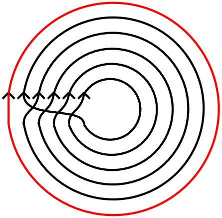
Question

How do we multiply these diagrams?

Multiplication of closed diagrams



$\alpha_{(5,1)}$

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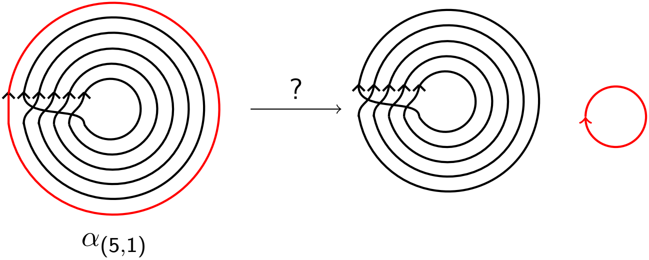

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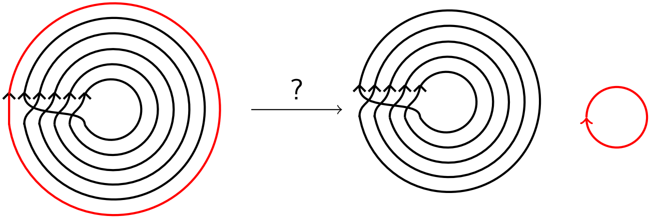

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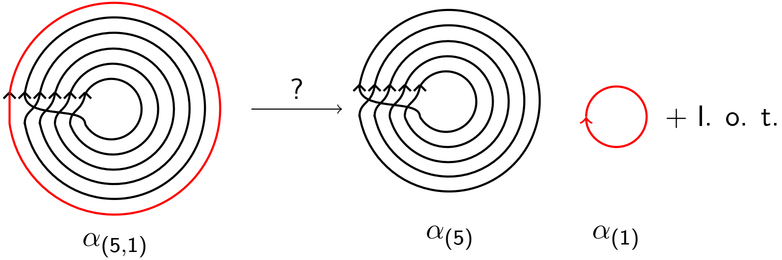

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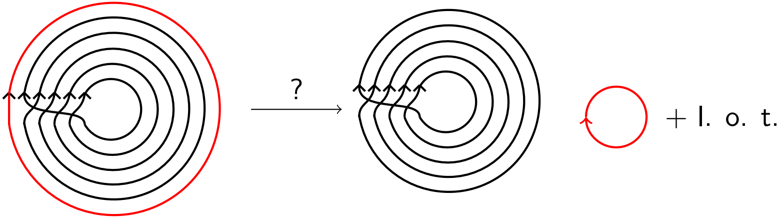


$\alpha_{(5,1)}$ $\alpha_{(5)}$ $\alpha_{(1)}$

$$\mathfrak{p}_{(5,1)} = \mathfrak{p}_{(5)}\mathfrak{p}_{(1)} + \text{lower order terms}$$

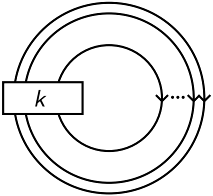
where $\{\mathfrak{p}_k\}$ is a non-homogeneous basis of $Sym = \Bbbk[p_1, p_2, \ldots]$.

Center of $\mathcal{H}_{tw}$

Theorem ('16, Kvigne, Licata, Mitchell)

There is an algebra isomorphism

$$\mathrm{End}_{\mathcal{H}}(\mathbb{1}) \simeq Sym^* = \Bbbk[\mathfrak{p}_1, \mathfrak{p}_2, \dots]$$



$$= \quad \alpha_k \quad \mapsto \quad \mathfrak{p}_k$$

Center of $\mathcal{H}_{tw}$

Theorem ('17, Kvigne, O., Reeks)

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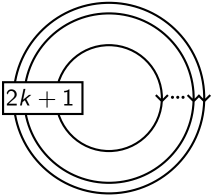
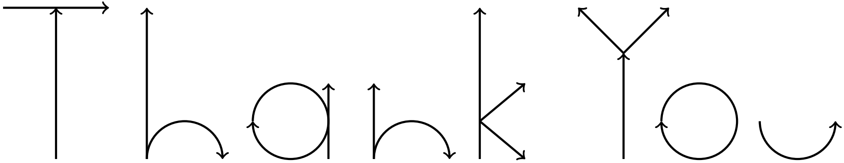


$$= \quad \alpha_{2k+1} \mapsto \mathfrak{p}_{2k+1}$$


Thank You

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